II YEAR - III SEMESTER
COURSE CODE: 7BMA3C2

CORE COURSE - VI – DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

Unit – I

Exact Differential Equations – Conditions for equation to be exact –Working rule for solving it – problems – Equations of the first order but of higher degree – Equations solvable for p, x, y, clairaut's form – Equations that do not contain (i) x explicitly (ii) y explicitly – Equations homogenous in x and y–Linear Equation with constant coefficients.

Unit – II

Linear equations with variable coefficients – Equations reducible to the linear equations – Simultaneous Differential Equations – First order and first degree – Simultaneous linear Differential Equations.

Unit – III

Linear equations of the second order – Complete Solution given a known integral – Reduction to Normal form – Change of the independent variable – Variation of parameters – Total Differential Equations – Necessary and Sufficient condition of integrability of $Pdx + Qdy + Rdz = 0$, Rule for solving it.

Unit – IV

Partial Differential Equations of the First oder – classifications of integrals – Derivations of Partial Differential Equations – Special methods – Standard forms – Charpit's method.

Unit – V

Flow of water from an Orifice – Falling bodies and other rate problems – Brachistochrone Problem – Tautochronous property of the Cycloid – Trajectories.

Text Book:

1. Differential Equations and its Applications by S.Narayanan&T.K.ManickavachagomPillay, S.Viswanathan (Printers& Publishers) Pvt. Ltd., 2015.

Unit I	Chapter 2 –sections 6.1 to 6.3; Chapter 4; Chapter5 –sections 1, 2, 3, 4
Unit II	Chapter 5–sections 5, 6; Chapter 6 – sections 1to 6
Unit III	Chapter 8–sections 1 to 4; Chapter 11
Unit IV	Chapter 12 – sections 1, 2, 3, 4, 5.1 to 5.4 & Section 6
Unit V	Chapter 3 – sections 2, 3, 4, 5; Chapter 10 – sections 1.1 – 1.3

Book for Reference:

1. Differential Equations and its Applications by Dr. S.Arumugam and Mr. A.ThangapandiIssac, New Gamma Publishing House, Palayamkottai, Edition, 2014.

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DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

SUBJECT CODE : 7BMA3C2

SYLLABUS :

UNIT - 1.

Exact differential equations - conditions for equation to be exact - working rule for solving problems - Equations of the first order but of higher degree - Eqns solvable for P, x, y, Clairaut's form - Eqns that do not contain,

(i) x explicity (ii) y explicity - eqns homogeneous in x and y - liniar eqn with constant coefficients.

UNIT - 2

Linear eqns with variable coefficients eqns reducible to be liniar eqns - simultaneously differential eqns - first order & first degree - simultaneous liniar differential eqns.

UNIT - 3

Linear eqns of the second order - complete solution given a known integral - Reduction to normal form - change of the dependent variable - variation of parameter - total differential eqns - necessary and coefficient condition of integrability of

$P\,dx + Q\,dy + R\,dz = 0$. rule for solving

UNIT - 4

Partial differential eqns of first order - classification of integrals - derivation of partial differential eqns - special method standard forms - Charpit's method.

UNIT - 5

Flow of water from an orifice - Falling bodies and other Rate problems - Brachistoch problems - Tautochronous - Properties of the cycloid - Trajectories.

TEXT BOOK :

1. Differential eqns & its Applications by S. Narayanan & T.K. Manickavasagam Pillai
2. Viswanathan (Printers & Publishers).

UNIT I	CHAPTER 2	SECTION 6.1 to 6.3
	CHPT 4, CHPT 5	SECTION 1, 2, 3, 4
UNIT II	CHPT 5	SECTION 5.6
	CHPT 6	SECTION 1 to 6
UNIT III	CHPT 8	SECTION 1 to 4
	CHPT 11	
UNIT IV	CHPT 12	SECTION 1 to 4, 5.1 to 5.4 & SECTION 6.
UNIT V	CHPT 3	SECTION 2 to 5
	CHPT 10	SECTION 1.1, to 1.3

UNIT - I

SECTION - 1 - EXACT DIFFERENTIAL EQUATIONS

6.1/20P An exact differential eqns is ~~some~~ obtained by equating an exact as, perfect differential to zero. we shall now investigate the condition that a given differential eqn may be exact and a method of integrating it. when the condition is satisfied

Let $Mdx + Ndy = 0$ be the differential eqn.

If this is exact, $Mdx + Ndy$ must have been obtained by derivating some function $u(x, y)$ & performing no other operation.

$$\therefore \; du = Mdx + Ndy.$$

$$\text{But } du = \frac{\delta u}{\delta x}dx + \frac{\delta u}{\delta y}dy.$$

Hence the necessary condition for the given eqn to be exact are

$$M = \frac{\delta u}{\delta x} \;\&\; N = \frac{\delta u}{\delta y}$$

$$\therefore \frac{\delta M}{\delta N} = \frac{\delta^2 y}{\delta y \delta x} = \frac{\delta^2 u}{\delta x \delta y} = \frac{\delta N}{\delta x}$$

Hence,

$\dfrac{\delta M}{\delta y} = \dfrac{\delta N}{\delta x}$. is the ~~condition~~ criterion for

$Mdx + Ndy = 0$ be the exact.

SECTION 2 : This condition is also sufficient.

If there be a function $u(x, y)$ where differential

$du = M dx + N dy$, we shall integrate relatively to x. As the partial differential $M dx$ could have been derived only from the terms containing x.

$$u = \int M dx + \text{terms not containing } x$$

$$= \int M dx + f(y) \quad \text{——— (1)}$$

Differentiating with respect to y partially,

$$\therefore \frac{\delta u}{\delta y} = \int \frac{\delta M}{\delta y} dx + f'(y)$$

As

$$N = \frac{\delta u}{\delta y}, \quad f'(y) = N - \int \frac{\delta M}{\delta y} dx \quad \text{——— (2)}$$

As the first member is independent of x, so is the second differentiation of this with respect to x leads us to,

$$\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y} = 0 \quad \text{(Hypothesis)}$$

Integrating (2) with r. to y

$$f(y) = \int \left\{ N - \int \frac{\delta M}{\delta y} dx \right\} dy + c \quad \text{——— (3)}$$

Where c is an arbitrary constant.

From (1)

$$u = \int M dx + \int \left\{ N - \int \frac{\delta M}{\delta y} dx \right\} dy + c$$

is a primitive required.

PROBLEMS :

PROBLEM : 1

$$\left(5x^4 + 3x^2y^2 - 2xy^3\right) dx + \left(2x^3y - 3x^2y^2 - 5y^4\right) dy$$

$$M = 5x^4 + 3x^2y^2 - 2xy^3$$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$$

$$N = 2x^3y - 3x^2y^2 - 5y^4$$

$$\frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given differential eqn is exact

Hence, solve the exact D.E $\int M dx + \int N dy = c$

$$\int \left(5x^4 + 3x^2y^2 - 2xy^3\right) dx + \int -5y^4 dy = 0$$

$$\left(\frac{5x^5}{5} + \frac{3y^2x^3}{3} - \frac{2x^2y^3}{2}\right) - \frac{5y^5}{5} = 0$$

$$x^5 + x^3y^2 - x^2y^3 - y^5 = 0$$

PROBLEM : 2

$$\left(e^y + 1\right) \cos x \, dx + e^y \sin x \, dy = 0$$

$$M = \left(e^y + 1\right) \cos x \, dx$$

$$= \left(e^y \cos x + \cos x\right) dx$$

$$\frac{\partial M}{\partial y} = e^y \cos x$$

$$N = e^y \sin x \, dy$$

$$\frac{\partial N}{\partial x} = e^y \cos x$$

The given D.E is exact,

Hence, solve the exact DE

$$\int M dx + \int N dy = c$$

$$\int (e^y+1)\cos x\,dx + \int e^y \sin x\,dy = 0$$

$$\int (e^y+1)\cos x\,dx + 0 = 0$$

$$\int e^y \cos x\,dx + \int \cos x\,dx = 0$$

$$e^y \sin x + \sin x = c$$

$$(e^y+1)\sin x = c.$$

PROBLEM :- 3

$$y(x^2+y^2+a^2)\frac{dy}{dx} + x(x^2+y^2-a^2) = 0$$

$$\frac{y(x^2+y^2+a^2)dy + x(x^2+y^2-a^2)dx}{dx} = 0$$

$$\underset{(Ndy)}{y(x^2+y^2+a^2)dy} + \underset{(Mdx)}{x(x^2+y^2-a^2)dx} = 0$$

$$M = x(x^2+y^2-a^2)$$

$$N = y(x^2+y^2+a^2)$$

$M = x(x^2+y^2-a^2)$ $\qquad$ $N = y(x^2+y^2+a^2)$

$M = x^3 + xy^2 - a^2x$ $\qquad$ $N = x^2y + y^3 + ya^2$

$\frac{\partial M}{\partial y} = 2xy$ $\qquad$ $\frac{\partial N}{\partial x} = 2xy$

the given DE is exact

Hence solve the exact DE

$$\int M dx + \int N dy = c$$

$$\int (x^3 + xy^2 - xa^2)\,dx + \int (y^3 + a^2y)\,dy$$

$$\frac{x^4}{4} + \frac{x^2}{2}y^2 - \frac{x^2}{2}a^2 + \frac{y^4}{4} + \frac{a^2y^2}{2} = c.$$

INTEGRATING FACTOR :-

If an equation of the form

$Mdx + Ndy = 0$ is not exact.

It can always be made exact by multiplication with a proper factor such a multiplies is called integrating factor.

For example,

$ydx - xdy = 0$ is not exact then we multiply by $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}$.

It will become a exact $\frac{1}{xy}, \frac{1}{x^2}, \frac{1}{y^2}$ are all integrating factor.

FORMULA :-

1. $d(xy) = ydx + xdy$

2. $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

3. $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

4. $d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{x^2+y^2}$

5. $d\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = \frac{ydx - xdy}{x^2+y^2}$

3. $(y^2e^x + 2xy)dx - x^2 dy = 0$

$$M = y^2e^x + 2xy \ dx$$

$$N = -x^2 dy$$

$$\frac{\partial M}{\partial y} = 2ye^x + 2x dx$$

$$\frac{\partial N}{\partial x} = -2x dy$$

$$d\left(\frac{x}{y}\right) = \frac{y dx - x dy}{y^2}$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

the given DE is not exact,

(~~Multiply by x^2~~ $\div$ by y^2)

$$\frac{y^2e^x dx}{y^2} + \frac{2xy \ dx - x^2 dy}{y^2} = 0$$

$$e^x dx + d\left(\frac{x^2}{y}\right) = 0$$

$$\int e^x dx + \frac{x^2}{y} = c$$

$$e^x + \frac{x^2}{y} = c.$$

1. Solve $\left(1 + e^{\frac{x}{y}}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

$$M = 1 + e^{\frac{x}{y}}$$

$$\frac{\partial M}{\partial y} = e^{x/y} \ x/y^2$$

$$N = e^{x/y}\left(1 - \frac{x}{y}\right)$$

$$= e^{x/y} - e^{x/y}\frac{x}{y}$$

$$\frac{\partial N}{\partial x} = e^{x/y} \ 1/y - \left[e^{x/y}\frac{1}{y} + x/y \ e^{x/y} \ 1/y\right]$$

$$= e^{x/y}\left[\frac{1}{y} - \frac{1}{y} - \frac{x}{y} \cdot \frac{1}{y}\right]$$

$= e^{\frac{x}{y}} \; x/y^2$

the given DE is exact,

Hence, solve the exact DE

$$\int M dx + \int N dy = c$$

$$\int (1 + e^{x/y})\, dx + 0 = c$$

$$\int dx + \int e^{x/y}\, dx = c$$

$$x + e^{x/y}\, \frac{x^2}{\cancel{2}y} = c$$

2. $(2xy + y - \tan y)\, dx + (x^2 - x\tan^2 y + \sec^2 y)\, dy = 0$

$$M = 2xy + y - \tan y$$

$1 - \sec^2 y = -\tan^2 y$

$$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y$$

$$= 2x - \tan^2 y$$

$$N = x^2 - x\tan^2 y + \sec^2 y$$

$$\frac{\partial N}{\partial x} = 2x - \tan^2 y$$

$1 - \sec^2 y = -\tan^2 y$

$\sec^2 y = \tan^2 y + 1$

$\tan^2 y = \sec^2 y$

The given DE is exact

Hence, solve the exact DE.

$$\int M dx + \int N dy = c$$

$$\int (2xy + y - \tan y)\, dx + \int (\sec^2 y)\, dy = c$$

$$\int 2xy\, dx + \int y\, dx - \int \tan y\, dx + \int \sec^2 y\, dy = c$$

$$\frac{\cancel{2}x^2}{\cancel{2}}\, y + xy - x\tan y + \tan y = c$$

$$x^2 y + xy - x\tan y + \tan y = c$$

4. $x\,dy - y\,dx = (x^2+y^2)\,dx$

$x\,dy - y\,dx - x^2 dx - y^2 dx = 0$

$(\div -)$

$-x\,dy + y\,dx + x^2 dx + y^2 dx = 0$

$dx\,(x^2+y^2+y) - x\,dy = 0$

$M\,dx + N\,dy = 0$

$M = (x^2+y^2+y)\,dx \qquad N = (-x)\,dy$

$\frac{\partial M}{\partial y} = 2y+1 \qquad \frac{\partial N}{\partial x} = -1$

The given D.E is not exact

Now, $x^2 dx + y^2 dx + y\,dx - x\,dy = 0$

(Multiply $\frac{1}{y^2}$)

$\frac{x^2}{y^2}\,dx + \frac{y^2}{y^2}\,dx + \frac{y}{y^2}\,dx - \frac{x}{y^2}\,dy = 0$

$\frac{y\,dx - x\,dy}{y^2} = \frac{x}{y}$

$\frac{x^2}{y^2}\,dx + \int dx + d\left(\frac{x}{y}\right) = 0$

$\frac{1}{y^2}\cdot\frac{x^3}{3} + x + \frac{x}{y} = c$

$\frac{x^3}{3y^2} + x + \frac{x}{y} = c.$

(3.) $\cos x \tan y + \cos(x+y)\,dx + \sin x \sec^2 y + \cos(x+y)\,dy$

$M = \cos x \tan y + \cos(x+y)$

$= \cos x \tan y + \cos x \cos y - \sin x \sin y$

$\frac{\partial M}{\partial y} = \cos x \sec^2 y - \cos x \sin y - \sin x \cos y$

$N = \sin x \sec^2 y + \cos(x+y)$

$= \sin x \sec y + \cos x \cos y - \sin x \sin y$

4. $x\,dy - y\,dx = (x^2+y^2)\,dx.$

$x\,dy - y\,dx - x^2dx - y^2dx = 0$

$(\div -)$

$-x\,dy + y\,dx + x^2dx + y^2dx = 0$

$dx\,(x^2+y^2+y) - x\,dy = 0$

$M\,dx + N\,dy = 0$

$M = (x^2+y^2+y)\,dx \qquad N = (-x)\,dy$

$\frac{\partial M}{\partial y} = 2y+1 \qquad \frac{\partial N}{\partial x} = -1$

The given D.E is not exact

Now, $x^2dx + y^2dx + y\,dx - x\,dy = 0$

(Multiply by $1/y^2$)

$\frac{x^2}{y^2}dx + \frac{y^2}{y^2}dx + \frac{y}{y^2}dx - \frac{x}{y^2}dy = 0$

$\frac{x^2}{y^2}dx + \int dx + d\left(\frac{x}{y}\right) = 0 \qquad \frac{y\,dx - x\,dy}{y^2} = \frac{x}{y}$

$\frac{1}{y^2}\cdot\frac{x^3}{3} + x + \frac{x}{y} = c$

$\frac{x^3}{3y^2} + x + \frac{x}{y} = c$

3. $\cos x\tan y + \cos(x+y)\,dx + \sin x\sec^2 y + \cos(x+y)\,dy$

$M = \cos x\tan y + \cos(x+y)$

$= \cos x\tan y + \cos x\cos y - \sin x\sin y$

$\frac{\partial M}{\partial y} = \cos x\sec^2 y - \cos x\sin y - \sin x\cos y$

$N = \sin x\sec^2 y + \cos(x+y)$

$= \sin x\sec^2 y + \cos x\cos y - \sin x\sin y$

$$\frac{\partial N}{\partial x} = \sec^2 y \cos x - \cos y \sin x - \sin y \cos x$$

6. $d\left[\log\left(\frac{x+y}{x-y}\right)\right] = 2\left[\frac{x\,dy - y\,dx}{x^2+y^2}\right]$

7. $d\left[\frac{x^2}{y}\right] = \frac{2yx\,dx - x^2\,dy}{y^2}$

8. $d\left[\frac{y^2}{x}\right] = \frac{2xy\,dy - y^2\,dx}{x^2}$

9. $d\left[\frac{1}{2}\log(x^2 \pm y^2)\right] = \frac{x\,dx \pm y\,dy}{x^2 \pm y^2}$

PROBLEMS :

1. $y\,dx - x\,dy - 3x^2y^2 e^{x^3} dx = 0$

Multiply by $\frac{1}{y^2}$

$$\frac{y\,dx - x\,dy}{y^2} - \frac{3x^2y^2 e^{x^3}}{y^2}\,dx = 0$$

$$\left[d\left(\frac{x}{y}\right)\right] - 3x^2 e^{x^3} dx = C$$

$$\int \frac{x}{y} - 3x^2 e^{x^3} dx = C$$

$$\frac{x}{y} - d\left[e^{x^3}\right] dx = C$$

$$\frac{x}{y} - e^{x^3} = C$$

$$\left[\int d\left(\frac{x}{y}\right) - \int d\left[e^{x^3}\right] dx \qquad \frac{x}{y} - e^{x^3} = C\right]$$

$\frac{\partial N}{\partial x} = \sec^2 y \cos x - \cos y \sin x - \sin y \cos x$

6. $d\left[\log\left(\frac{x+y}{x-y}\right)\right] = 2\left[\frac{x\,dy - y\,dx}{x^2+y^2}\right]$

7. $d\left[\frac{x^2}{y}\right] = \frac{2yx\,dx - x^2\,dy}{y^2}$

8. $d\left[\frac{y^2}{x}\right] = \frac{2xy\,dy - y^2\,dx}{x^2}$

9. $d\left[\frac{1}{2}\log(x^2 \pm y^2)\right] = \frac{x\,dx \pm y\,dy}{x^2 \pm y^2}$

PROBLEMS :

1. $y\,dx - x\,dy - 3x^2y^2 e^{x^3} dx = 0$

Multiply by $\frac{1}{y^2}$

$\frac{y\,dx - x\,dy}{y^2} - \frac{3x^2y^2 e^{x^3}}{y^2}\,dx = 0$

$\left[d\left(\frac{x}{y}\right)\right] - 3x^2 e^{x^3} dx = C$

$\int \frac{x}{y} - 3x^2 e^{x^3} dx = c$

$\frac{x}{y} - d\left[e^{x^3}\right] dx = c$

$\frac{x}{y} - e^{x^3} = c$

$\left[\int d\left(\frac{x}{y}\right) - \int d\left[e^{x^3}\right] dx\right.$

$\left.\frac{x}{y} - e^{x^3} = c\right]$

Rules for finding Integrating Factor

1. When $Mx + Ny \neq 0$ & the eqn is homogenous

$\frac{1}{Mx+Ny}$ is an integrating factor of

$M\,dx + N\,dy = 0$

2. When $Mx - Ny = 0$ & the eqn is of the form

$f_1(x,y)\,y\,dx + f_2(x,y)\,x\,dy = 0$

$\frac{1}{Mx-Ny}$ is an integrating factor.

3. If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone say

$f(x)$ then integrating factor in $\int f(x)\,dx$

4. If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone say

$f(y)$ then integrating factor in $\int f(y)\,dy$

Problems :

$\frac{y^3 - 3x^2y}{x^4 - y^4}$

1. solve $(x^3 - 3xy^2)\,dx - (y^3 - 3x^2y)\,dy = 0$ $\frac{y^3}{x^4 - y^4}$

$(x^3 - 3xy^2)\,dx - (y^3 - 3x^2y)\,dy = 0$

$M = x^3 - 3xy^2 \qquad N = (y^3 - 3x^2y)$

This type is homogenous eqn

$Mx = x^4 - 3x^2y^2 \qquad Ny = -y^4 + 3x^2y^2$

$Mx = x^4 - 3x^2y^2 \quad -y^4 + 3x^2y^2$

$= x^4 - y^4 \neq 0$

$\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$

$\frac{\partial M}{\partial y} = -6xy$

$\frac{\partial N}{\partial x} = 6y$

$\frac{1}{Mx+Ny}$ is an I.F

$\Rightarrow \frac{1}{x^4-y^4}$

(×ply the DE by $\frac{1}{x^4-y^4}$)

$$\frac{x^3-3xy^2}{x^4-y^4}\,dx - \frac{(y^3-3x^2y)}{x^4-y^4}\,dy$$

an exact DE

$$\therefore \underbrace{\int M\,dx}_{\text{constant}} + \underbrace{\int N\,dy}_{\text{Not having } x \text{ term}} = C$$

$$\int\left(\frac{x^3-3xy^2}{x^4-y^4}\right)dx + 0 = 0$$

$$\int \frac{x^3}{x^4-y^4}\,dx - 3\int \frac{xy^2}{x^4-y^4}\,dx = C$$

$$= \frac{1}{4}\left[\int \frac{dx}{x} = \log x\right]$$

$$\left[\int \frac{1}{x^4-y^4} \quad \frac{4x^3}{x^4-y^4} = \text{There is no } 4, \text{ so } \times \text{ \& } \div \text{ by } 4\right]$$

$$= \frac{1}{4}\int \frac{4x^3}{x^4-y^4}\,dx - 3\int \frac{xy^2}{(x^2-y^2)(x^2+y^2)}\,dx = C$$

$$= \frac{1}{4}\log(x^4-y^4) - 3\int \frac{xy^2}{(x^2+y^2)(x^2-y^2)}\,dx = C$$

$$\int \frac{xy^2}{(x^2+y^2)(x^2-y^2)}\,dx \rightarrow (1)$$

Put $u = x^2$

$du = 2x\,dx$

$x\,dx = \frac{du}{2}$

① $\Rightarrow$

$$\frac{1}{4}\log(x^4-y^4) - 3y^2\int \frac{\frac{1}{2}du}{(u^2+y^2)(u-y^2)} = c$$

$$\frac{1}{4}\log(x^4-y^4) - \frac{3y^2}{2}\int\left[\frac{du}{\int u^2-(y^2)^2}\right] = c$$

$$\frac{1}{4}\log(x^4-y^4) - \frac{3}{2}y^2\left[\frac{1}{2y^2}\log\left(\frac{u-y^2}{u+y^2}\right)\right] = c$$

$$\left[\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)\right]$$

$$\frac{1}{4}\log(x^4-y^4) - \frac{3}{4}\frac{y^2}{y^2}\left[\log\left(\frac{u-y^2}{u+y^2}\right)\right] = c$$

$$\frac{1}{4}\log(x^4-y^4) - \frac{3}{4}\left[\log\left(\frac{u-y^2}{u+y^2}\right)\right] = c$$

$u = x^2$

$$\Rightarrow \frac{1}{4}\log(x^4-y^4) - \frac{3}{4}\left[\log\left(\frac{x^2-y^2}{x^2+y^2}\right)\right] = c$$

$$\log(x^4-y^4) - 3\left[\log\left(\frac{x^2-y^2}{x^2+y^2}\right)\right] = c$$

2. Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

$$\left(\frac{\partial M}{\partial y} = x^2 - 4xy \qquad \frac{\partial N}{\partial x} = -3x^2 + 6xy \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$$

$M = x^2y - 2xy^2$

$N = -(x^3 - 3x^2y)$

This type is homoginous eqn.

$Mx = x^3y - 2x^2y^2 \qquad Ny = -x^3y + 3x^2y^2$

$\cancel{Mx + Ny = \frac{x^3}{y} - 2x^2y^2 - \frac{x^3}{y} + 3x^2y^2}$

$Mx + Ny = \cancel{x^3}\ x^3y - 2x^2y^2 - x^3y + 3x^2y^2$

① $\Rightarrow$

$$\frac{1}{4}\log(x^4-y^4) - 3y^2\int\frac{\frac{1}{2}du}{(u^2+y^2)(u-y^2)} = c$$

$$\frac{1}{4}\log(x^4-y^4) - \frac{3y^2}{2}\int\left[\frac{du}{\int u^2-(y^2)^2}\right] = c$$

$$\frac{1}{4}\log(x^4-y^4) - \frac{3}{2}y^2\left[\frac{1}{2y^2}\log\left(\frac{u-y^2}{u+y^2}\right)\right] = c.$$

$$\left[\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)\right]$$

$$\frac{1}{4}\log(x^4-y^4) - \frac{3}{4}\frac{y^2}{y^2}\left[\log\left(\frac{u-y^2}{u+y^2}\right)\right] = c$$

$$\frac{1}{4}\log(x^4-y^4) - \frac{3}{4}\left[\log\left(\frac{u-y^2}{u+y^2}\right)\right] = c$$

$u = x^2$

$$\Rightarrow \frac{1}{4}\log(x^4-y^4) - \frac{3}{4}\left[\log\left(\frac{x^2-y^2}{x^2+y^2}\right)\right] = c.$$

$$\log(x^4-y^4) - 3\left[\log\left(\frac{x^2-y^2}{x^2+y^2}\right)\right] = c$$

2. Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

$M = x^2y - 2xy^2$

$N = -(x^3 - 3x^2y)$

$$\left[\begin{array}{l} \frac{\partial M}{\partial y} = x^2 - 4xy \\ \frac{\partial N}{\partial x} = 3x^2 + 6xy \\ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \end{array}\right]$$

This type is homogenous eqn.

$Mx = x^3y - 2x^2y^2$ $\qquad Ny = -x^3y + 3x^2y^2$

~~$Mx + Ny = \frac{x^3}{y} - 2x^2y^2 - \frac{x^3}{y} + 3x^2y^2$~~

$Mx + Ny = \cancel{x^3}\ x^3y - 2x^2y^2 - x^3y + 3x^2y^2$

$= x^2y^2 \neq 0$

$\therefore \frac{1}{Mx+Ny}$ is an IF

$\Rightarrow \frac{1}{x^2y^2}$

$\times$ ply the DE by $\frac{1}{x^2y^2}$

$\left(\frac{x^2y - 2xy^2}{x^2y^2}\right)dx - \left(\frac{x^3 - 3x^2y}{x^2y^2}\right)dy$, an exact DE

$\therefore \int\limits_{\text{const } y} M\,dx + \int\limits_{\text{Not having } x} N\,dy = c$

$\left[\frac{x^3 - 3x^2y}{x^2y^2} \quad \frac{x^3}{x^2y^2} - \frac{3x^2y}{x^2y^2} \quad \frac{x}{y^2} - \frac{3}{y} \rightarrow \text{not having } x \right.$

$\int \frac{x^2y - 2xy^2}{x^2y^2}dx + 3\int \frac{1}{y}dy = c$

$\int \frac{y(x^2 - 2xy)}{x^2y^2}dx + 3\int \frac{1}{y}dy = c$

$\int \frac{x^2 - 2xy}{x^2y}dx + 3\int \frac{1}{y}dy = c$

$\int \frac{x^2}{x^2y}dx - 2\int \frac{xy}{x^2y}dx + 3\int \frac{1}{y}dy = c$

$\int \frac{1}{y}dx - 2\int \frac{1}{x}dx + 3\int \frac{1}{y}dy = c$

$\frac{x}{y} - 2\log x + 3\log y = c$

$\frac{x}{y} - \log x^2 + \log y^3 = c$

$\frac{x}{y} + \log y^3 - \log x^2 = c$

$\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = c$

3 solve $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = c$

$M = y^3 - 2yx^2 \qquad N = 2xy^2 - x^3$

$\frac{\partial M}{\partial y} = 3y^2 - 2x^2 \qquad \frac{\partial N}{\partial x} = 2y^2 - 3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

The given DE is not exact

$Mx = xy^3 - 2yx^3$

$Ny = 2xy^3 - yx^3$

$Mx + Ndy = xy^3 - 2yx^3 + 2xy^3 - yx^3 \quad (+)$

$= 3xy^3 - 3x^3y \neq 0$

$\frac{1}{Mx+Ny}$ is an I.F

$\frac{1}{3xy^3 - 3yx^3} = \frac{1}{3xy(y^2-x^2)}$

$\frac{1}{xy(y^2-x^2)}$ is an I.F

[Multiply by $\frac{1}{xy(y^2-x^2)}$ the D.E]

$\frac{y^3 - 2yx^2}{xy(y^2-x^2)}dx + \frac{2xy^2 - x^3}{xy(y^2-x^2)}dy = 0$

(Take y as com)

$\frac{y(y^2-2x^2)}{yx(y^2-x^2)}dx + \frac{x(2y^2-x^2)}{xy(y^2-x^2)}dy = 0$

$\frac{y^2-2x^2}{x(y^2-x^2)}dx + \frac{(2y^2-x^2)}{y(y^2-x^2)}dy = 0$

$\frac{y^2-x^2-x^2}{x(y^2-x^2)}dx + \frac{y^2+y^2-x^2}{y(y^2-x^2)}dy = 0$

$$\left[\frac{\cancel{y^2-x^2}}{x\cancel{(y^2-x^2)}} - \frac{x^2}{x(y^2-x^2)}\right]dx + \left[\frac{y^2}{y(y^2-x^2)} + \frac{y^2-x^2}{y(y^2-x^2)}\right]dy = 0$$

$$\left(\frac{1}{x} - \frac{x}{y^2-x^2}\right)dx + \left(\frac{y}{y^2-x^2} + \frac{1}{y}\right)dy = 0$$

$$\int M dx + \int N dy = c$$

$(\div \times \text{ by } 2)$

$$\int \frac{1}{x}dx + \frac{1}{2}\int \frac{x}{y^2-x^2}dx + \int \frac{1}{y}dy = c$$

$$\log x + \frac{1}{2}\log(y^2-x^2) + \log y = \log c$$

$$2\log x + \log(y^2-x^2) + 2\log y = \log c$$

$$\log x^2 + \log(y^2-x^2) + \log y^2 = \log c$$

$$\log(x^2y^2)(y^2-x^2) = \log c$$

$$x^2y^2(y^2-x^2) = 0$$

PRACTICAL RULE FOR SOLVING AN EXACT D.E.

Integrate Mdx as if y were constant and those terms in $N\,dy$ that do not give the terms already occurring.

The sum of these integrals equated to a constant gives the soln.

EXAMPLE 1 / 21 P.

Solve $(a^2 - 2xy - y^2)\,dx - (x+y)^2\,dy = 0$.

Soln :-

The given D.E is

$$(a^2 - 2xy - y^2)\,dx - (x+y)^2\,dy = 0 \longrightarrow ①$$

① is of the form,

$$Mdx + N\,dy = 0.$$

Here,

$$M = a^2 - 2xy - y^2$$
$$N = -(x+y)^2$$

$$\frac{\partial M}{\partial y} = -2x - 2y$$
$$= -2(x+y)$$

$$\frac{\partial N}{\partial x} = -2(x+y)$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence ① is an exact D.E

$$\int Mdx = \int (a^2 - 2xy - y^2)\,dx$$

$$= a^2 - 2y\frac{x^2}{2} - y^2 x$$

$\int N$ (not containing x) $dy = \int -y^2\,dy$

$$= \frac{-y^3}{3}$$

the soln is

$$\int M dx + \int N \text{ (not containing } x) \, dy = c$$

$$x^2 - x^2y - xy^2 - \frac{y^3}{3} = c \text{ is the Req. eqn.}$$

EXAMPLE : 2

solve $(2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x})dy$

$+ (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)\,dx = 0.$

<u>Soln.</u>

$$M dx + N dy = 0.$$

$$M = 12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y$$

$$\frac{\partial M}{\partial y} = 12x^2 + 2x2y + 0 - 4\times 3y^2 + 2e^{2x} - e^y$$

$$= 12x^2 + 4xy - 12y^2 + 2e^{2x} - e^y.$$

$$N = 2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x}$$

$$\frac{\partial N}{\partial x} = 2y\times 2x + 4\times 3x^2 - 12y^2 - e^y + 2e^{2x}$$

$$= 4xy + 12x^2 - 12y^2 - e^y + 2e^{2x}.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int M dx = \int (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)\,dx$$

$$= 12y\frac{x^3}{3} + 2y^2\frac{x^2}{2} + \frac{4x^4}{4} - 4y^3x + 2ye^{2x} - e^{y}x$$

$= 4x^3y + x^2y^2 + x^4 - 4y^3x + ye^{2x} - e^{y}x$

$\int N \text{ (not containing } x) \, dy = \int 3y^2 \, dy$

$= 3\dfrac{y^3}{3}$

$= y^3$

FORMULAS :

(i) $d(xy) = y\,dx + x\,dy$

(ii) $d\left(\dfrac{x}{y}\right) = \dfrac{y\,dx - x\,dy}{y^2}$

(iii) $d\left(\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2}$

(iv) $d\left(\tan^{-1}\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2+y^2}$

(v) $d\left(\tan^{-1}\dfrac{x}{y}\right) = \dfrac{y\,dx - x\,dy}{x^2+y^2}$

(vi) $d\left[\log\dfrac{x+y}{x-y}\right] = 2 \cdot \dfrac{x\,dy - y\,dx}{x^2-y^2}$

$\int M\,dx = a^2 - 2y\dfrac{x^2}{2} - y^2x$

$\int N \text{ (not containing } x) \, dy = \int -y^2 \, dy$

$= -\dfrac{y^3}{3}$

The solution is

$\int M\,dx + \int N \text{ (not containing } x) \, dy = c$

$a^2 - x^2y + xy^2 - \frac{y^3}{3} = c$ is the required.

(vii) $d\left(\frac{x^2}{y}\right) = \frac{2y\,x\,dx - x^2dy}{y^2}$

(viii) $d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$

(ix) $d\left[\frac{1}{2}\log(x^2 \pm y^2)\right] = \frac{xdx \pm ydy}{x^2 \pm y^2}$

EXAMPLE : 1 / 23P

solve $a(xdy + 2ydx) = xy\,dy$.

If we divide by xy, ($\frac{1}{xy}$ is an integrating facto

$$a\left(\frac{dy}{y} + 2\frac{dx}{x}\right) = dy$$

$$a\log(yx^2) = y + c.$$

H.W EXAMPLE : 2 / 23P

Solve : $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0.$

EXAMPLE : 3 / 23P

Solve $(y^2e^x + 2xy)dx - x^2dy = 0$

soln :-

$M = (y^2e^x + 2xy)dx$ $\qquad N = -x^2dy$

$\frac{\partial M}{\partial y} = 2ye^x + 2xdx$ $\qquad \frac{\partial N}{\partial x} = -2x\,dy$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The given DE is not exact

(÷ by y^2) given eqn

$$\frac{y^2 e^x\, dx}{y^2} + \underbrace{\frac{2xy\, dx}{y^2} - \frac{x^2 dy}{y^2}} = 0.$$

$$e^x dx + d\left(\frac{x^2}{y}\right) = 0 \qquad \left[\because d\left(\frac{x^2}{y}\right) = \frac{2yx\,dx - x^2 dy}{y^2}\right]$$

$$\int e^x dx + \left(\frac{x^2}{y}\right) = c$$

$$e^x + \frac{x^2}{y} = c.$$

Hence solved.

Rules for Finding Integrating Factors :

(24 / P.)

1, When $Mx + Ny \neq 0$, and the eqn is homogeneous $\frac{1}{Mx+Ny}$ is an integrating factor of $Mdx + Ndy = 0$.

2, when $Mx - Ny \neq 0$, and the eqn is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$, $\frac{1}{Mx-Ny}$ is an integrating factors.

3, If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone say $f(x)$, then the integrating factor is $e^{\int f(x)\, dx}$

4, If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone say $f(y)$, then the integrating factor $e^{\int f(y)\, dy}$

Ex : 1 / 24 P.

Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

soln :

($\div$ by y^2) given eqn.

$$\frac{y^2 e^x\,dx}{y^2} + \underbrace{\frac{2xy\,dx}{y^2} - \frac{x^2 dy}{y^2}} = 0.$$

$$e^x dx + d\left(\frac{x^2}{y}\right) = 0 \qquad \left[\because d\left(\frac{x^2}{y}\right) = \frac{2y\,x\,dx - x^2 dy}{y^2}\right]$$

$$\int e^x dx + \left(\frac{x^2}{y}\right) = c$$

$$e^x + \frac{x^2}{y} = c$$

Hence solved.

RULES FOR FINDING INTEGRATING FACTORS :

(24 / P.)

1, When $Mx + Ny \neq 0$, and the eqn is homogeneous $\frac{1}{Mx+Ny}$ is an integrating factor of $Mdx + Ndy = 0$.

2, When $Mx - Ny \neq 0$, and the eqn is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$, $\frac{1}{Mx - Ny}$ is an integrating factors.

3, If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone say $f(x)$, then the integrating factor is $e^{\int f(x)dx}$

4, If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone say $f(y)$, then the integrating factor. $e^{\int f(y)dy}$

Ex :1 /24P.

Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

soln :

$M = x^2y - 2xy^2$ $\quad N = -(x^3 - 3x^2y)$

this type is homogenous eqn.

$Mx = x^3y - 2x^2y^2$ $\quad Ny = -x^3y + 3x^2y^2$

$Mx + Ny = \frac{x^3}{y} - 2x^2y^2 - \frac{x^3}{y} + 3x^2y^2$

$= x^2y^2$

$Mx + Ny \neq 0$

$\frac{1}{Mx + Ny}$ is an I.F

$\Rightarrow \frac{1}{x^2y^2}$ is I.F

the D.E by $\frac{1}{x^2y^2}$

$\left(\frac{x^2y - 2xy^2}{x^2y^2}\right)dx - \left(\frac{x^3 - 3x^2y}{x^2y^2}\right)dy$ is an exact D.E

$\int M dx + \int N dy = c$ — term contains y term so (x)

$\Rightarrow \int \frac{x^2y - 2xy^2}{x^2y^2} dx + 3\int \frac{1}{y} dy = c.$

$\Rightarrow \int \frac{y(x^2 - 2xy)}{x^2y^2} dx + 3\int \frac{1}{y} dy = c$

$\Rightarrow \int \frac{x^2 - 2xy}{x^2y} dx + 3\int \frac{1}{y} dy = c.$

$\Rightarrow \int \frac{x^2}{x^2y} dx - 2\int \frac{xy}{x^2y} dx + 3\int \frac{1}{y} dy = c$

$\Rightarrow \int \frac{1}{y} dx - 2\int \frac{1}{x} dx + 3\int \frac{1}{y} dy = c.$

$\Rightarrow \frac{x}{y} - 2\log x + 3\log y = c$

$$\Rightarrow \frac{x}{y} - \log x^2 + \log y^3 = c$$

$$\Rightarrow \frac{x}{y} + \log y^3 - \log x^2 = c$$

$\log m - \log n = \log \frac{m}{n}$

$$\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = c.$$

Hence solved.

Ex : 3/ 25P

solve $(y - 3x^2)dx - x(1 - xy^2)dy = 0.$

soln :-

the eqn can be written in the form,

$$ydx - 3x^2dx - x\,dy + x^2y^2dy = 0$$

$$ydx - x\,dy - 3x^2dx + x^2y^2dy = 0.$$

($\div$ by x^2)

$$\frac{ydx - xdy}{x^2} - \frac{3x^2dx}{x^2} + \frac{x^2y^2dy}{x^2} = 0$$

(by formula)

$$-d\left(\frac{y}{x}\right) - d(3x) + d(y^2) = 0$$

$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

Int - $\int$ this eqn.

$$-\frac{y}{x} - 3x + \frac{y^3}{3} = c$$

Hence Solved.

Ex : 4 26/P

solve $\frac{dy}{dx} = \frac{2x}{x^2 + y^2 - 2y}$

soln :

the eqn can be written in the form

$$(x^2+y^2-2y)\,dy = 2x\,dx$$

$$(x^2+y^2)\,dy - 2y\,dy = 2x\,dx$$

$$(x^2+y^2)\,dy = 2y\,dy + 2x\,dx$$

$$dy = \frac{d(y^2+x^2)}{x^2+y^2}$$

$$dy = d(\log)(x^2+y^2)$$

$$\therefore y = \log(x^2+y^2) + c.$$

Hence Solved.

Ex : 15 26/P.

Solve $(1+xy^2)\,dx + (1+x^2y)\,dy = 0$

soln :

The eqn can be written in the form,

$$dx + xy^2\,dx + dy + x^2y\,dy = 0$$

$$dx + dy + xy^2\,dx + x^2y\,dy = 0$$

$$d(x+y) + xy(y\,dx + x\,dy) = 0$$

$$d(x+y) + xy\,d(xy) = 0$$

$$d(x+y) + \frac{1}{2}\,d(xy)^2 = 0$$

Integrating this eqn,

$$\therefore x+y+\frac{1}{2}(xy)^2 = C$$

Hence Solved.

Ex : 6 26/P

solve $(x^2+y^2)(x\,dx + y\,dy) = a^2(x\,dy = y\,dx)$

soln :

the eqn can be written in the form,

$$x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{x^2+y^2}$$

formula

$$\left(d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2+y^2} \right)$$

$$\frac{1}{2} d(x^2+y^2) = a^2 d \tan^{-1}\left(\frac{y}{x}\right)$$

Integrating this eqn,

$$\therefore \frac{1}{2}(x^2+y^2) = a^2 \tan^{-1}\left(\frac{y}{x}\right) + c$$

Ex : 6 / 26P

solve $(x^2+y^2)(x\,dx + y\,dy) = a^2(x\,dy - y\,dx)$

soln :-

this eqn can be written in the form.

$$x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{x^2+y^2}$$

$$\frac{1}{2} d(x^2+y^2) = a^2 d \tan^{-1}\left(\frac{y}{x}\right)$$

Integrating this eqn,

$$\therefore \frac{1}{2}(x^2+y^2) = a^2 \tan^{-1}\left(\frac{y}{x}\right) + c$$

Hence Solved.

12/8/2020

CHAPTER - IV

(60/P)

EQUATIONS OF THE FIRST ORDER BUT OF HIGHER DEGREE.

TYPE A :-

Eqns solvable for $\frac{dy}{dx}$

we shall denote $\frac{dy}{dx}$ here after by P.

Let the eqn of the first order and of the n^{th} degree is $P^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_n = 0$ → ①

where P_1, P_2, P_n denote functions of x & y

suppose the 1st member of ① can be Resolved into functions the 1st degree of the form

$(P - R_1)\ (P - R_2)\ (P - R_3) \dots (P - R_n)$

Any relation b/w x & y which makes any of these functions vanish is a soln of ①,

Let the primitives of $P - R_1 = 0$, $P - R_2 = 0$ etc be $\phi_1(x, y, c_1) = 0$, $\phi_2(x, y, c_2) = 0 \dots$ $\phi_n(x, y, c_n) = 0$ Respectively.

where c_1, c_2, c_n are Arbitary Constants.

without any loss of generality, we can replace $c_1, c_2 \ldots c_n$ by c. where c is arbitary constant.

Hence, the soln of ① is,

$$\phi_1(x, y, c_1)\, \phi_2(x, y, c_2) \ldots \phi_n(x, y, c_n) = 0.$$

EXAMPLE ① 60/P

solve $x^2p^2 + 3xyp + 2y^2 = 0$.

soln :-

given diff eqn is $x^2p^2 + 3xyp + 2y^2 = 0$

$a = x^2 \quad b = 3yx \quad c = 2y^2$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3yx \pm \sqrt{9y^2x^2 - 8x^2y^2}}{2x^2}$$

$$= \frac{-3yx \pm \sqrt{y^2x^2}}{2x^2}$$

$$= \frac{-3xy \pm xy}{2x^2}$$

$$= \frac{-3xy + xy}{2x^2}$$

$$= -\frac{2xy}{2x^2}$$

$$= \frac{-y}{x}$$

$$P = \frac{-3yx - yx}{2x^2}$$

$$= \frac{-4yx}{2x^2}$$

$$= \frac{-2y}{x}$$

$P = \frac{-y}{x}$

$\frac{dy}{dx} = \frac{-y}{x}$

$\int \frac{dy}{y} = -\int \frac{dx}{x}$

$\log y = -\log x + \log c$

$\log y + \log x = \log c$

$\log(xy) = \log c$

$xy = c$

$P = -\frac{2y}{x}$

$\frac{dy}{dx} = -\frac{2y}{x}$

$\int \frac{dy}{2y} = -\int \frac{dx}{x}$

$\frac{1}{2}\log y = -\log x + \log c$

$\frac{1}{2}\log y + \log x = \log c$

$\log y^{1/2} + \log x = \log c$

$\log x^2 y = \log c$

$x^2 y = c$

The soln is $(xy - c)(x^2 y - c) = 0$.

Ex : 2/ 61 P

solve $p^2 + \left(x + y - \frac{2y}{x}\right)p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0$

soln :-

Given diff eqn is

$$p^2+\left(x+y-\frac{2y}{x}\right)p+xy+\frac{y^2}{x^2}-y-\frac{y^2}{x}=0.$$

$$a=1 \quad b=x+y-\frac{2y}{x} \quad c=xy+\frac{y^2}{x^2}-y-\frac{y^2}{x}$$

$$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-\left(x+y-\frac{2y}{x}\right)\pm\sqrt{\left(x+y-\frac{2y}{x}\right)^2-4\left(xy+\frac{y^2}{x^2}-y-\frac{y^2}{x}\right)}}{2}$$

$$=\frac{-\left(x+y-\frac{2y}{x}\right)\pm\sqrt{(x+y)^2+\frac{4y^2}{x^2}-2(x+y)\left(\frac{2y}{x}\right)-4xy-\frac{4y^2}{x^2}+4y+\frac{4y^2}{x}}}{2}$$

$$=\frac{-(x+y)-\frac{2y}{x})\pm\sqrt{x^2+y^2+2xy+\frac{4y^2}{x^2}-\frac{4xy}{x}-\frac{4y^2}{x}-4xy-\frac{4y^2}{x^2}+4y+\frac{4y^2}{x}}}{2}$$

$$=\frac{-\left(x+y-\frac{2y}{x}\right)\pm\sqrt{x^2+y^2-2xy}}{2}$$

$$p=\frac{-\left(x+y-\frac{2y}{x}\right)\pm(x-y)}{2}$$

case(i)

$$\Rightarrow \frac{1}{2}\left[-x-y+\frac{2y}{x}+x-y\right]$$

$$=\frac{1}{2}\left[-2y+\frac{2y}{x}\right]$$

$$=\frac{2}{2}\left[-y+\frac{y}{x}\right]$$

$$=-y+\frac{y}{x}$$

$$p=\frac{y}{x}-y$$

$$p=\frac{y}{x}-y$$

$$\frac{dy}{dx}=\frac{y}{x}-y$$

$$\frac{dy}{dx}=y\left(\frac{1}{x}-1\right)$$

$$\frac{dy}{y}=\left(\frac{1}{x}-1\right)dx$$

$$\int\frac{dy}{y}=\int\left(\frac{1}{x}-1\right)dx$$

$$\log y=\log x-x+\log c$$

$$\log y-\log x=-x+\log c$$

$$\log\frac{y}{x}=-x+\log c$$

$$\frac{y}{x}=-xe^{c}$$

$$y/x=ce^{-x} \quad y=cxe^{-x}$$

case(ii)

$$p=\frac{-\left(x+y-\frac{2y}{x}\right)-(x-y)}{2}$$

$$=\frac{-x-y+\frac{2y}{x}-x+y}{2}$$

$$=\frac{1}{2}\left[-2x+\frac{2y}{x}\right]$$

$$=\frac{2}{2}\left[-x+\frac{y}{x}\right]$$

$$p=\frac{y}{x}-x$$

$$\frac{dy}{dx}=\frac{y}{x}-x$$

$$\int\frac{dy}{dx}=\int\frac{1}{x}-x\,dx$$

$$\log y=\log x-x\,dx+c$$

$$\frac{y}{x}=-x+c$$

$$y+x^2-cx=0$$

The gen. soln is $(y-cxe^{-x})$

$$(y+x^2-cx)=0$$

1. solve : $p^2 - 5p + 6 = 0$

soln :-

given diff eqn is $p^2 - 5p + 6 = 0$

$$(p-3)(p-2) = 0$$

$$p-3 = 0 \qquad p-2 = 0$$

$$p = 3 \qquad p = 2$$

case (i)

when $p = 3$

$$\frac{dy}{dx} = 3$$

$$dy = 3dx$$

$$\int dy = \int 3dx$$

$$y = 3x + c$$

$$y - 3x - c = 0$$

case (ii)

when $p = 2$

$$\frac{dy}{dx} = 2$$

$$dy = 2dx$$

$$\int dy = \int 2dx$$

$$y = 2x + c$$

$$y - 2x - c = 0$$

The combined eqn soln is

$$(y - 3x - c)(y - 2x - c) = 0$$

2. solve $x^2p^2 + xyp - 6y^2 = 0$

Ex 3.0 sum

soln :-

given diff eqn is $x^2p^2 + xyp - 6y^2 = 0$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = x^2$$

$$b = xy$$

$$c = -6y^2$$

$$= \frac{-xy \pm \sqrt{x^2y^2 - 4(x^2)(-6y^2)}}{2x^2}$$

$$= \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2} = \frac{-xy \pm \sqrt{25x^2y^2}}{2x^2}$$

$$= \frac{-xy \pm 5xy}{2x^2}$$

(Take x as common & cancel) $\frac{\cancel{x}(y \pm 5y)}{2x^{\cancel{2}}}$

$$P = \frac{-y \pm 5y}{2x}$$

$$P = \frac{-y+5y}{2x} = \frac{4y}{2x}$$

$$P = \frac{2y}{x}$$

$$P = \frac{-y-5y}{2x} = \frac{-6y}{2x}$$

$$P = \frac{-3y}{x}$$

case(i)

when $P = \frac{2y}{x}$

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$dy = \frac{2y}{x}dx$$

$$\frac{dy}{2y} = \frac{dx}{x}$$

$$\int \frac{dy}{2y} = \int \frac{dx}{x}$$

$$\frac{1}{2}\log y = \log x + c$$

$$\frac{1}{2}\log y - \log x - c = 0$$

(mult by 2)

(logm - logn)

$$\log y - \underline{2\log x} - 2c = 0$$

$\downarrow$ constant ($\log x^2$)

$$\log\left(\frac{y}{x^2}\right) - c = 0$$

$$\left(\frac{y}{x^2} - c\right) = 0$$

case(ii)

when $P = \frac{-3y}{x}$

$$\frac{dy}{dx} = \frac{-3y}{x}$$

$$\frac{dy}{3y} = \frac{-dx}{x}$$

$$\int \frac{dy}{3y} = -\int \frac{dx}{x}$$

$$\frac{1}{3}\log y = -\log x + c$$

$$\frac{1}{3}\log y + \log x - c = 0$$

($\times$ by 3)

$$\log y + 3\log x - 3c = 0$$

$$\log y + \log x^3 - 3c = 0$$

$$\log x^3y - c = 0$$

$$x^3y - c = 0$$

the solution is $(y-cx^2)(x^3y-c)=0$

4. solve $p^2y+p(x-y)-x=0$

<u>soln</u>:-

$$p^2y+p(x-y)-x=0$$

given eqn is $p^2y+p(x-y)-x=0$

$$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$a=y$

$b=(x-y)$

$c=-x$

$$=\frac{-(x-y)\pm\sqrt{(x-y)^2+4xy}}{2y}$$

$$=\frac{-(x-y)\pm\sqrt{x^2+y^2-2xy+4xy}}{2y}$$

$$=\frac{-x+y\pm\sqrt{x^2+y^2+2xy}}{2y}$$

$(a+b)^2=a^2+b^2+2ab$

$\sqrt{x^2+y^2}=x+y$

$$p=\frac{-x+y\pm x+y}{2y}$$

$$p=\frac{-x+y+x+y}{2y}=\frac{2y}{2y}=1$$

$$p=\frac{-x+y-x-y}{2y}=\frac{-2x}{2y}=\frac{-x}{y}$$

when $P = 1$

$$\frac{dy}{dx} = 1$$

$$\int dy = \int dx$$

$$y = x + c$$

$$x - y + c = 0$$

when $P = \frac{-x}{y}$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + c$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$$\Rightarrow x^2 + y^2 = c.$$

the soln is $(x - y + c)(x^2 + y^2 - c) = 0.$

24.12.2020 EXERCISE III / 13^{th} sum / 66 P.

(4) solve:

$$P^2 + 2yp\cot x = y^2$$

Soln :-

given eqn $P^2 + 2yp\cot x - y^2 = 0$

$a = 1$, $b = 2y\cot x$, $c = -y^2$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x - 4(1)(-y^2)}}{2(1)}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2(\cot^2 x + 1)}}{2}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\operatorname{cosec}^2 x}}{2}$$

$$= \frac{-2y\cot x \pm 2y\operatorname{cosec} x}{2}$$

$$= -y\frac{\cancel{2}(\cot x \pm \operatorname{cosec} x)}{\cancel{2}}$$

$$P = -y(\cot x \pm \operatorname{cosec} x)$$

case (i)

$$P = -y(\cot x + \operatorname{cosec} x)$$

$$\frac{dy}{dx} = -y(\cot x + \operatorname{cosec} x)$$

$$\frac{dy}{y} = -dx(\cot x + \operatorname{cosec} x)$$

$$\frac{dy}{y} = -\left(\frac{\cos x}{\sin x} + \operatorname{cosec} x\right)dx$$

$$\int\frac{dy}{y} = -\int\left(\frac{\cos x}{\sin x} + \operatorname{cosec} x\right)dx$$

$$\log y = -\left(\log \sin x - \log(\operatorname{cosec} x \cot x)\right)$$

$$y + y\cos x = c$$

$$y(1 + \cos x) = c$$

case (ii)

$$y(1 - \cos x) = c$$

$$\int \frac{1}{\sin x}\cos x + \int \operatorname{cosec} x$$

$$\frac{1}{\sin}(-\sin x) + \cos x(\operatorname{cosec} x$$

$\log \sin x$

the combined soln is $y(1 \pm \cos x) = c$

is the req. soln.

Exer x III / 30 / 67 P.

solve. $x^2p^2 + xyp - 6y^2 = 0$

soln : $a = x^2$, $b = xy$, $c = -6y^2$

Lyn diff eqn is $x^2p^2 + xyp - 6y^2 = 0$

$$p = \frac{-xy \pm \sqrt{x^2y^2 - 4x^2(-6y^2)}}{2x^2}$$

$$= \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2}$$

$$= \frac{-xy \pm 5xy}{2x^2}$$

$$P = \frac{-y \pm 5y}{2x}$$

case (i)

$$P = \frac{-y+5y}{2x}$$
$$= \frac{4y}{2x}$$
$$= \frac{2y}{x}$$

where $P = \frac{2y}{x}$

$$\frac{dy}{dx} = \frac{2y}{x}$$
$$dy = \frac{2y}{x}dx$$
$$\frac{dy}{2y} = \frac{dx}{x}$$
$$\int \frac{dy}{2y} = \int \frac{dx}{x}$$
$$\frac{1}{2}\log y = \log x + c$$
$$\frac{1}{2}\log y - \log x - c = 0$$

($\times$ by 2)

$$\log y - 2\log x - 2c = 0$$
$$\log y - \log x^2 - c = 0$$
$$\log\left(\frac{y}{x^2}\right) - c = 0$$
$$\left(\frac{y}{x^2} - c\right) = 0$$

case (ii)

$$P = \frac{-y-5y}{2x}$$
$$= \frac{-6y}{2x}$$
$$= \frac{-3y}{x}$$

when $P = -\frac{3y}{x}$

$$\frac{dy}{dx} = \frac{-3y}{x}$$
$$dy = \frac{-3y}{x}dx$$
$$\frac{dy}{3y} = -\frac{dx}{x}$$
$$\int \frac{dy}{3y} = -\int \frac{dx}{x}$$
$$\frac{1}{3}\log y = -\log x + c$$
$$\frac{1}{3}\log y + \log x - c = 0$$

($\times$ by 3)

$$\log y + 3\log x - 3c = 0$$
$$\log y + \log x^3 - 3c = 0$$
$$\log x^3 y - c = 0$$
$$x^3 y - c = 0$$

The combined soln is

$(y - cx^2)(x^3 y - c) = 0$ is the Req soln.

Exer/12/66P

solve $y - 2px = x^2p^4$ (eqns solvable for y)

soln:

The gn diff eqn is $y - 2px = x^2p^4$

which is of $y = f(x,p)$ $\quad y = 2px + x^2p^4 \longrightarrow (1)$

diff (1) w.r. to x

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4x^2p^3\frac{dp}{dx}$$

$$p = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4x^2p^3\frac{dp}{dx}$$

$$p + 2xp^4 + 2x(1+2xp^3)\frac{dp}{dx} = 0$$

$$p(1+2xp^3) + (1+2xp^3)2x\frac{dp}{dx} = 0$$

$$(1+2xp^3)(p + 2x\cdot\frac{dp}{dx}) = 0$$

$$1 + 2xp^3 = 0 \quad ; \quad p + 2x\cdot\frac{dp}{dx} = 0$$

case (i)

$1 + 2xp^3 = 0$

$2xp^3 = -1$

$p^3 = \frac{-1}{2x} \rightarrow (2)$

sub (2) in (1)

$y = 2px + x^2p^4$

$= px(2 + xp^3)$

$= px\left(2 + x\left(\frac{-1}{2x}\right)\right)$

$= px\left(2 + \left(\frac{-1}{2}\right)\right)$

$= px\left(\frac{3}{2}\right)$

$y = \frac{3}{2}(px)$

(taking cube) $(\times by\ 3)$

$y^3 = \frac{27}{8}p^3x^3$

case (ii)

$p + 2x\frac{dp}{dx} = 0$

$2x\frac{dp}{dx} = -p$

$2\frac{dp}{p} = -\frac{dx}{x}$

$2\int\frac{dp}{p} = -\int\frac{dx}{x}$

$2\log p = -\log x + \log c$

$\log p^2 + \log x = \log c$

$\log xp^2 = \log c$

$\boxed{xp^2 = c} \longrightarrow (3)$

$$y^3 = \frac{27}{8}\left(\frac{-1}{2x}\right)x^3$$

$$y^3 = \frac{-27}{16}x^2$$

$$16y^3 = -27x^2$$

given $y = 2px + x^2p^4 \longrightarrow ①$

eliminate p from ① & ②

$$y - x^2p^4 = 2px$$

(Taking square)

$$(y - x^2p^4)^2 = 4p^2x^2$$

$\boxed{p^2 = c}$

$$(y - c^2)^2 = 4x(p^2x)$$

$(y-c)^2 = 4xc$ is the general soln of ①

9/3/2020 CLAIRAUT'S FORM :-

the equation of the form

$$y = px + f(p) \longrightarrow ①$$

is known as clairauts equation.

the solution of ① is $y = cx + f(c) \rightarrow ②$

XIII. 9. $e^{3x}(p-1) + p^3e^{2y} = 0$. solve it.

Exer

soln:-

given diff eqn $e^{3x}(p-1) + p^3e^{2y} = 0 \longrightarrow ①$

Put $x = e^{x}$, $y = e^{y}$

$dx = e^x dx$ $\qquad dy = \frac{dy}{e^y}$

$dx = \frac{dx}{e^x}$ $\qquad dy = dy/y$

$$\frac{dy}{dx} = \frac{dy}{e^y} \cdot \frac{e^x}{dx}$$

$$\frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx}$$

② $\Rightarrow \int e^{\int (x+1)(x-1)^2} = \int (x+1)(x-1)^2 \, du$

Let $u = x+1$ $\quad \int dv = \int (x-1)^2$

$du = dx$, $\quad v = \frac{(x-1)^3}{3}$

$$z = (x+1)\frac{(x-1)^3}{3} - \int \frac{(x-1)^3}{3} \, dx.$$

$$= \frac{(x+1)(x-1)^3}{3} - \frac{1}{3}\frac{(x-1)^4}{4}$$

$$= \frac{4(x+1)(x-1)^3 - (x-1)^4}{12}$$

$$= \frac{(x-1)^3}{12}\left[4(x+1) - (x-1)\right]$$

$$= \frac{(x-1)^3}{12}\left[4x+4-x+1\right]$$

$$z = \frac{(x-1)^3(3x+5)}{12}$$

$$\frac{dz}{dx} = \frac{1}{12}\left[(x-1)^3(3) + (3x+5)\,3\,(x-1)^2\right]$$

$$= \frac{3(x-1)^2}{12}\left[(x-1) + 3x+5\right]$$

$$= \frac{(x-1)^2}{4}(4x+4)$$

$$= \frac{(x-1)^2\, 4(x+1)}{4}$$

$$\frac{dz}{dx} = (x-1)^2(x+1)$$

$$\frac{d^2z}{dx^2} = (x-1)^2(1) + (x+1)2(x-1)$$

$$= (x-1)\left[(x-1) + 2(x+1)\right]$$

$$= (x-1)\left[x-1+2x+2\right]$$

$$\frac{d^2z}{dx^2} = (x-1)(3x+1)$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P_1\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{(x-1)(3x+1) + 1\left(\frac{-3x+1}{x^2-1}\right)\left((x-1)^2(x+1)\right)}{\left[(x-1)^2(x+1)\right]^2}$$

$$= \frac{(x-1)(3x+1) - \frac{3x+1}{(x+1)(x-1)}(x-1)^2(x+1)}{\left[(x-1)^2(x+1)\right]^2}$$

$$= \frac{(x-1)(3x+1) - 3x-1)(x-1)}{\left[(x-1)^2(x+1)\right]^2}$$

$$P_1 = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{\left[\frac{6(x+1)}{(x-1)(3x+5)}\right]}{\left[(x-1)^2(x+1)\right]^2}$$

$$= \frac{36(x+1)^2}{(x+1)^2(x-1)^4(x-1)^2(3x+5)^2}$$

$$Q_1 = \frac{36}{(x-1)^6(3x+5)^2}$$

(3) $\Rightarrow$

$$\frac{d^2y}{dz^2} + \frac{36}{(x-1)^6(3x+5)^2}y = 0$$

$$\frac{d^2y}{dz^2} + \frac{36y}{\left[(x-1)^3(3x+5)\right]^2} = 0 \qquad z = \frac{(x-1)^3(3x+5)}{12}$$

$$\frac{d^2y}{dz^2} + \frac{36y}{(12z)^2} = 0 \qquad 12z = (x-1)^3(3x+$$

$$\frac{d^2y}{dx^2} + \frac{y}{4x^2} = 0$$

$$4x^2 \cdot \frac{d^2y}{dx^2} + y = 0$$

let $u = \log x$; $D = \frac{d}{du}$ & $x = e^u$

$$(4x^2D^2+1)y = 0$$

To find CF

$$(4x^2D^2+1)y = 0$$

$$4\left[\theta(\theta-1)+1\right]y = 0$$

$$(4\theta^2 - 4\theta + 1)y = 0$$

The AE is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)(2m-1) = 0$$

$$m = \frac{1}{2}, \frac{1}{2}.$$

$$CF = \left[c_1 u + c_2\right] e^{\frac{1}{2}}$$

$$= \left[c_1 \log x + c_2\right] x^{\frac{1}{2}}$$

$$y = \left[c_1 \log \frac{(x-1)^3(3x+5)}{12} + c_2\right]\left[\frac{(x-1)^3(3x+5)}{12}\right]$$

H.W

1. $(D^2+1)y = \tan^2 x \longrightarrow (1)$

$(D^2+1)y = 0 \longrightarrow (2)$

soln of (2) is $y = CF$

To find CF

$$(D^2+1)y = 0$$

The AE is

$$m^2+1 = 0$$

$$m^2 = -1$$

$$\boxed{m = \pm i}$$

$y = A\cos x + B\sin x$, $A, B \rightarrow$ constant we assume that

$y = A(x)\cos x + B(x)\sin x \longrightarrow$ (3) is the sum of (1)

where, $A(x)$ & $B(x)$ such that

$A_1\cos x + B_1\sin x = 0 \longrightarrow$ (4)

$A_1\frac{d}{dx}\cos x + B_1\frac{d}{dx}\sin x = \tan x$

$-A_1\sin x + B_1\cos x = \tan^2 x \longrightarrow$ (5)

solving (4) & (5),

$$\begin{aligned} A_1\cos x\sin x + B_1\sin^2 x &= 0 \\ -A_1\cos x\sin x + B_1\cos^2 x &= \tan^2 x\cdot\cos x \\ \hline B_1(\sin^2 x + \cos^2 x) &= \tan^2 x\cos x \end{aligned}$$

$$B_1 = \tan^2 x\cdot\cos x$$

$$\frac{dB}{dx} = \tan^2 x\cos x$$

$$dB = \tan^2 x\cdot\cos x\,dx$$

$$\int dB = \int\tan^2 x\cdot\cos x\cdot dx$$

$$\int dB = \int\frac{\sin^2 x}{\cos^2 x}\cos x\cdot dx$$

$$B = \int\frac{\sin^2 x}{\cos x}dx$$

$$= \int\frac{1-\cos^2 x}{\cos x}dx.$$

$$= \int\frac{1}{\cos x}dx - \int\frac{\cos^2 x}{\cos x}dx$$

$$= \int\sec x\,dx - \int\cos x\,dx$$

$$B = \log(\sec x + \tan x) - \sin x + C_2$$

the value B_1 subs in (4),

$$A_1\cos x + (\tan^2 x\cos x)\sin x = 0$$

$$A_1\cos x + \frac{\sin^2 x}{\cos^2 x}\cdot\cos x\sin x = 0$$

$$A_1 \cos x + \frac{\sin^3 x}{\cos x} = 0$$

$$\cos x \, A_1 = -\frac{\sin^3 x}{\cos x}$$

$$A_1 = \frac{-\sin^3 x}{\cos^2 x}$$

$$A_1 = -\tan^2 x \cdot \sin x$$

$$\frac{dA}{dx} = -\tan^2 x \cdot \sin x$$

$$dA = -\tan^2 x \sin x \cdot dx.$$

1. solve $yy'' = y'^2 - y'$

<u>soln</u>:

<u>G.T</u> $$y \cdot \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 - \left(\frac{dy}{dx}\right) \rightarrow (1)$$

$$y \cdot \frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{dy}{dx} - 1\right)$$

this eqn is not containing x directly

Let $\frac{dy}{dx} = P$ Then $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dy}\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \frac{dP}{dy} \cdot P.$$

(1) $\Rightarrow$ $$yP\frac{dP}{dy} = P^2 - P$$

$$y \cdot \frac{dP}{dy} = \frac{P^2 - P}{P}$$

$$y \cdot \frac{dP}{dy} = \frac{P(P-1)}{P}$$

$$y \cdot \frac{dP}{dy} = P - 1$$

$$\frac{dP}{P-1} = \frac{dy}{y}$$

$$= \frac{-dP}{1-P} = \frac{dy}{y}$$

$$\frac{dy}{y} + \frac{dP}{1-P} = 0$$

Integrating,

$$-\log(1-P) + \log y = \log c$$

$$\log(1-P)^{-1} + \log y = \log c$$

$$\log y(1-P)^{-1} = \log c$$

$$y(1-P)^{-1} = c$$

$$\frac{y}{1-P} = c$$

$$1-P = \frac{y}{c}$$

$$P = 1 - \frac{y}{c}$$

$$\frac{dy}{dx} = \frac{c-y}{c}$$

$$\frac{dy}{c-y} = \frac{dx}{c}$$

$$-\log(c-y) = \frac{1}{c}x + c_1$$

$$\frac{1}{c}x + \log(c-y) + c_1 = 0 \,//$$

TOTAL DIFFERENTIAL EQNS :-

Eqns of the type $\boxed{P\,dx + Q\,dy + R\,dz = 0}$ where P, Q, R are the functions of x, y, z are called total diff. eqns.

NECESSARY CONDITION FOR THE INTEGRABILITY :

$$\begin{vmatrix} P & Q & R \\ P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = 0$$

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

⇒ which is the necessary condition for the integrability for the eqn.

$$Pdx + Qdy + Rdz = 0$$

RULE FOR SOLVING $\boxed{Pdx + Qdy + Rdz = 0}$

METHOD 1:

Taking one variable constant

PROBLM

1. verify the condition of integrability & solve $(y+z)\,dx + dy + dz = 0$

soln:

G.T. $(y+z)dx + dy + dz = 0 \longrightarrow (1)$

This is of the type $Pdx + Qdy + Rdz = 0$

$P = y+z \qquad Q = 1 \qquad R = 1$

$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial R}{\partial x} = 0$

$\frac{\partial P}{\partial z} = 1 \qquad \frac{\partial Q}{\partial z} = 0 \qquad \frac{\partial R}{\partial y} = 0$

we have the cdn of integrability is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$$\Rightarrow (y+z)(0-0) + 1(0-1) + 1(1-0) = 0$$

$$0 - 1 + 1 \Rightarrow 0$$

∴ cdn of integrability is satisfied.

we assume that, x = a constant

then, $dx = 0$

the gn diff eqn $\Rightarrow dy + dz = 0$

Integrating, $\cancel{dy + dz}$

$y + z$ = constant which is independent of y & z

$y + z = \phi(x)$

Differentiating $dy + dz = du \longrightarrow$ (2)

compare (1) & (2)

$(y+z)\,dx + d\phi = 0$

$\phi(x)\,dx + d\phi = 0$

$d\phi = -\phi\,dx$

$\frac{d\phi}{\phi} = -dx$

$\int \frac{d\phi}{\phi} = \int -dx$

$\log\phi + x = c_1$

$\log\phi = -x + c_1$

$\phi = e^{-x+c_1}$

$= e^{-x}\, e^{c_1}$

$\phi = ce^{-x}$ where $c = e^{c_1}$

$y + z = ce^{-x}$ is the soln of (1)

2. solve $(y+z)dx + (z+x)dy + (x+y)dz = 0$

<u>soln</u>

$(y+z)dx + (z+x)dy + (x+y)dz = 0 \longrightarrow$ (1)

This is of the type $Pdx + Qdy + Rdz = 0$.

$P = y+z \qquad Q = z+x \qquad R = (x+y)$

$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 1 \qquad \frac{\partial R}{\partial x} = 1$

$\frac{\partial P}{\partial z} = 1 \qquad \frac{\partial Q}{\partial z} = 1 \qquad \frac{\partial R}{\partial y} = 1$

we have cdn of integrability is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$$y+z\,(1-1) + z+x\,(1-1) + (x+y\,(1-1) = 0$$

$= 0$

$\therefore$ Cnd of integrability is Verified.

we assume that $z = a$ constant

Then $dz = 0$

the gn diff eqn $\Rightarrow (y+z)dx + (z+x)dy = 0$

$$(y+z)\,dx = -(z+x)\,dy$$

$$\frac{dx}{z+x} = -\frac{dy}{y+z}$$

Integrating,

$$(y+z)dx = -(z+x)dy$$

$$\frac{dx}{z+x} = -\frac{dy}{y+z}$$

$$\frac{dx}{z+x} + \frac{dy}{y+z} = 0$$

Integrating,

$$\log(z+x) + \log(y+z) = \log c$$

$$\log(z+x)(y+z) = \log c$$

$$(z+x)(y+z) = c$$

$$(z+x)(y+z) = \phi(z)$$

Differentiating

$$(z+x)(dy+dz) + (y+z)(dx+dz) = \phi'(z)\,dz$$

$$(z+x)\,dy + (z+x)\,dz + (y+z)dx + (y+z)\,dz = \phi' dz$$

$$(y+z)dx + (z+x)dy + (z+x+y+z-\phi')\,dz = 0$$

$$(y+z)dx + (x+z)\,dy + (x+y+2z-\phi')dz = 0 \longrightarrow (2)$$

comparing (1) & (2),

$$-\phi' + 2z = 0$$

$$\phi' = 2z$$

$$d\phi = 2z$$

Integrating, $\phi = z^2 + c$

$$(z+x)(y+z) = z^2 + c$$

$$yz + z^2 + xy + xz - z^2 = c$$

$xy + yz + xz = c$ is soln of (1)

Pg : 213

1. solve $(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0$

soln :

G.T $(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0 \longrightarrow (1)$

this is of type $Pdx + Qdy + Rdz = 0$

$P = y^2+yz \qquad Q = xz+z^2 \qquad R = y^2-xy$

$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = z \qquad \frac{\partial R}{\partial x} = -y$

$\frac{\partial P}{\partial z} = y \qquad \frac{\partial Q}{\partial z} = x+2z \qquad \frac{\partial R}{\partial y} = 2y - x$

we have the cdn of integrability,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$\Rightarrow y^2+yz(-2y) + xz+z^2(-y-y) + (y^2-xy)(2y+\cancel{z}-z)$

$\Rightarrow zy^2 - 2y^3 + xy^2 + yz^2 - 2y^2z + xyz - 2yxz - 2yz^2 + 2y^3$
$+ zy^2 - zy^2 - xy^2y - xyz + 2y^3 - 2y^2x$

$= 0$

$\therefore$ cnd of integrability is verified.

we assume that $z =$ a constant.

then, $dz = 0$.

the gn diff eqn $\Rightarrow (y^2+yz)dx + (xz+z^2)dy =$

$(y^2+yz)dx = -(xz+z^2)dy$

$$\frac{dx}{xz+z^2} = -\frac{dy}{y^2+yz}$$

$$\frac{dx}{z(x+z)} = -\frac{dy}{y(y+z)}$$

$$\frac{dx}{(x+z)} = \frac{-z\,dy}{y(y+z)} = 0 \longrightarrow (2)$$

Integrating, $\log(x+z)$

consider

$$\frac{z}{y(y+z)}\,dy = \frac{A}{y} + \frac{B}{y+z}\,dy$$

$$z = A(y+z) + By$$

Put, $\boxed{y=-z}$

$$z = A(0) + Bz$$

$$z = -Bz$$

$$\boxed{z = -1}$$

Put $\boxed{y=0}$

$$z = A(z) + 0$$

$$\boxed{A = 1}$$

$$\frac{dy}{y} - \frac{dy}{y+z}$$

$$\log y - \log(y+z) = 0$$

$$\log y + \log(y+z)^{-1} = 0$$

$$\log y\,(y+z)^{-1} = 0$$

$$\log \frac{y}{y+z} = 0$$

(2) $\Rightarrow \log(x+z) + \log\frac{y}{y+z} = \log c$

$$\frac{(x+z)y}{y+z} = c - \phi(z) \longrightarrow (3)$$

Diff, (2)

$$\frac{(y+z)\left[(dx+dz)y + (x+z)dy\right] - y(x+z)(dy+dz)}{(y+z)^2} = \phi'(z)dz$$

$$(y^2+yz)\,dx + (xz+z^2)\,dy + dz\left[y^2 - xy - \phi'(x)\,(y+z)^2\right] = 0 \longrightarrow (4)$$

comparing (1) & (4)

$$\phi'(y+z)^2\,dz = 0$$

$$(y+z)^2\,dz \neq 0$$

$$\phi'(x) = 0$$

Integrating, $\phi = c_1$ constant

$$\frac{(x+z)y}{y+z} = c$$

$(x+z)y = c(y+c)$ is the soln of (1)

METHOD 2 :

Rearranging the terms

PROBLEM :-

EG :

solve $(y+z)\,dx + dy + dz = 0$

soln

G.T $(y+z)\,dx + dy + dz = 0$

$P = y+z \qquad Q = 1 \qquad R = 1$

$$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial R}{\partial x} = 0$$

$$\frac{\partial P}{\partial z} = 1 \qquad \frac{\partial Q}{\partial z} = 0 \qquad \frac{\partial R}{\partial y} = 0$$

we have cdn of integrability is,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$$(y+z)(0-0) + 1(0-1) + 1(1-0)$$

$$0 - 1 + 1 \Rightarrow 0$$

$\therefore$ verified

$$(y+z)\,dx = -dy - dz$$

$$dx = -\frac{(dy+dz)}{y+z}$$

$$dx + \frac{dy+dz}{y+z} = 0$$

Integrating,

$$\int dx + \int \frac{d(y+z)}{y+z} = 0$$

$$x + \log(y+z) = 0$$

$$\log(y+z) = -x + c$$

$$y+z = e^{-x} + c$$

$$y+z = e^{-x} e^{c}$$

$y+z = e^{-x} c_1$ is soln of ①

H.W

1. $(y-z)\,dx + (z-x)\,dy + (x-y)\,dz = 0$

soln

G.T

$$(y-z)\,dx + (z-x)\,dy + (x-y)\,dz = 0$$

this is of the type $P\,dx + Q\,dy + R\,dz = 0$

$P = y - z$ $\qquad Q = z - x$ $\qquad R = x - y$

$\frac{\partial P}{\partial y} = 1$ $\qquad \frac{\partial Q}{\partial x} = -1$ $\qquad \frac{\partial R}{\partial x} = 1$

$\frac{\partial P}{\partial z} = -1$ $\qquad \frac{\partial Q}{\partial z} = 1$ $\qquad \frac{\partial R}{\partial y} = -1$

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (y-z)(1+1) + (z-1)(1+1) + (x-y)(1+1)$$

$$= 2y - z + 2z - 2a + 2y - 2y$$

$$= 0$$

$\therefore$ Verified

we assume that z = a constant

then $dz = 0$

① $\Rightarrow (y-z)\,dx + (z-x)\,dy = 0.$

$(y-z)dx = -(z-x)dy$

$$\frac{dx}{z-x} = -\frac{dy}{y-z}$$

integrating,

$$\log(z-x) = -\log(y-z)\log c$$

$$\log\frac{y-z}{z-x} = \log c$$

$$\frac{y-z}{z-x} = \phi(z)$$

Differentiating,

$$\frac{(z-x)(dy-dz) - (y-z)(dz-dx)}{(z-x)^2} = \phi(z)dz$$

$$\frac{zdy - zdz + xdz - xdy - [y\,dz - ydx - zdz + zdx]}{(z-x)^2} = \phi'(z)dz$$

$$\{(y-z)dx + (z-x)dy + (x-y)dz = -\phi'(z) - x^2\}dz = 0$$

comparing ① & ②

$$[x-y-p'(z)(z-x)^2] = 0.$$

$$-\phi'(z-x)^2 = 0$$

integrating,

$$\frac{y-z}{z-x} = c$$

$y-z = c(z-x)$ which is a Required soln

1. solve : $p^2 - 5p + 6 = 0$

soln :-

given diff eqn is $p^2 - 5p + 6 = 0$

$$(p-3)(p-2) = 0$$

$p-3 = 0$ $\qquad$ $p-2 = 0$

$p = 3$ $\qquad$ $p = 2$

case (i)

when $p = 3$

$$\frac{dy}{dx} = 3$$

$$dy = 3dx$$

$$\int dy = \int 3dx$$

$$y = 3x + c$$

$$y - 3x - c = 0$$

case (ii)

when $p = 2$

$$\frac{dy}{dx} = 2$$

$$dy = 2dx$$

$$\int dy = \int 2dx$$

$$y = 2x + c$$

$$y - 2x - c = 0$$

the combined eqn soln is

$$(y - 3x - c)(y - 2x - c) = 0$$

2. solve $x^2p^2 + xyp - 6y^2 = 0$

soln :-

given diff eqn is $x^2p^2 + xyp - 6y^2 = 0$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = x^2$

$b = xy$

$c = -6y^2$

$$= \frac{-xy \pm \sqrt{x^2y^2 - 4(x^2)(-6y^2)}}{2x^2}$$

$= \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2} \quad = \frac{-xy \pm \sqrt{25x^2y^2}}{2x^2}$

$= \frac{-xy \pm 5xy}{2x^2}$ (Take x as common & cancel) $\frac{x(y \pm 5y)}{2x^2}$

$P = \frac{-y \pm 5y}{2x}$

$P = \frac{-y + 5y}{2x}$

$= \frac{4y}{2x}$

$P = \frac{2y}{x}$

$P = \frac{-y - 5y}{2x}$

$= \frac{-6y}{2x}$

$P = \frac{-3y}{x}$

case (i)

when $P = \frac{2y}{x}$

$\frac{dy}{dx} = \frac{2y}{x}$

$dy = \frac{2y}{x} dx$

$\frac{dy}{2y} = \frac{dx}{x}$

$\int \frac{dy}{2y} = \int \frac{dx}{x}$

$\frac{1}{2} \log y = \log x + c$

$\frac{1}{2} \log y - \log x - c = 0$

(mult by 2)

(logm - logn)

$\log y - 2\log x - 2c = 0$

↓ constant ($\log x^2$)

$\log \left(\frac{y}{x^2}\right) - c = 0$

$\left(\frac{y}{x^2} - c\right) = 0$

case (ii)

when $P = \frac{-3y}{x}$

$\frac{dy}{dx} = \frac{-3y}{x}$

$\frac{dy}{3y} = \frac{-dx}{x}$

$\int \frac{dy}{3y} = -\int \frac{dx}{x}$

$\frac{1}{3} \log y = -\log x + c$

$\frac{1}{3} \log y + \log x - c = 0$

($\times$ by 3)

$\log y + 3\log x - 3c = 0$

$\log y + \log x^3 - 3c = 0$

$\log x^3 y - c = 0$

$x^3 y - c = 0$

the solution is $(y - cx^2)(x^3 y - c) = 0$

4. solve $p^2 y + p(x-y) - x = 0$

<u>soln</u> :-

$$p^2 y + p(x-y) - x = 0$$

given eqn is $p^2 y + p(x-y) - x = 0$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = \cancel{xy}\ y$

$b = (x-y)$

$c = -\cancel{y}\ x$

$$= \frac{-(x-y) \pm \sqrt{(x-y)^2 + 4xy}}{2y}$$

$$= \frac{-(x-y) \pm \sqrt{x^2 + y^2 - 2xy + 4xy}}{2y}$$

$$= \frac{-x+y \pm \sqrt{x^2+y^2+2xy}}{2y}$$

$(a+b)^2 = a^2 + b^2 + 2ab$

$\sqrt{x^2+y^2} = x+y$

$$p = \frac{-x+y \pm x+y}{2y}$$

$$p = \frac{-x+y+x+y}{2y} = \frac{2y}{2y} = 1$$

$$p = \frac{-x+y-x-y}{2y} = \frac{-2x}{2y} = \frac{-x}{y}$$

when $P = 1$

$$\frac{dy}{dx} = 1$$

$$\int dy = \int dx$$

$$y = x + c$$

$$x - y + c = 0$$

when $P = \frac{-x}{y}$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + c$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$$\Rightarrow x^2 + y^2 = c.$$

The soln is $(x - y + c)(x^2 + y^2 - c) = 0$.

21-8-2020 EXERCISE III / 13^{th} sum / 66 P.

(4) solve:

$$P^2 + 2yp\cot x = y^2$$

Soln:-

given eqn $P^2 + 2yp\cot x - y^2 = 0$

$a = 1$, $b = 2y\cot x$, $c = -y^2$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x - 4(1)(-y^2)}}{2(1)}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2(\cot^2 x + 1)}}{2}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\operatorname{cosec}^2 x}}{2}$$

$$= \frac{-2y\cot x \pm 2y\operatorname{cosec} x}{2}$$

$$= -y\frac{\cancel{2}(\cot x \pm \operatorname{cosec} x)}{\cancel{2}}$$

$P = -y(\cot x \pm \operatorname{cosec} x)$

case (i)

$P = -y(\cot x + \operatorname{cosec} x)$

$\frac{dy}{dx} = -y(\cot x + \operatorname{cosec} x)$

$\frac{dy}{y} = -dx(\cot x + \operatorname{cosec} x)$

$\frac{dy}{y} = -\left(\frac{\cos x}{\sin x} + \operatorname{cosec} x\right)dx$

$\int \frac{dy}{y} = -\int \left(\frac{\cos x}{\sin x} + \operatorname{cosec} x\right)dx$

$\log y = -(\log \sin x - \log(\operatorname{cosec} x \cot x))$

$y + y\cos x = c$

$y(1 + \cos x) = c$

case (ii)

$y(1 - \cos x) = c$

$\int \frac{1}{\sin x} \cos x + \int \operatorname{cosec} x$

$\frac{1}{\sin}(-\sin x) + \cos x(\operatorname{cosec} x)$

$\log \sin x$

the combined soln is $y(1 \pm \cos x) = c$ is the req. soln.

Exer x III / 30 / 67 P

solve. $x^2p^2 + xyp - 6y^2 = 0$

soln :

Lyn diff eqn is $x^2p^2 + xyp - 6y^2 = 0$

$$P = \frac{-xy \pm \sqrt{x^2y^2 - 4x^2(-6y^2)}}{2x^2}$$

$$= \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2}$$

$$= \frac{-xy \pm 5xy}{2x^2}$$

$$P = \frac{-y \pm 5y}{2x}$$

case (i)

$$P = \frac{-y+5y}{2x}$$
$$= \frac{4y}{2x}$$
$$= \frac{2y}{x}$$

where $P = \frac{2y}{x}$

$$\frac{dy}{dx} = \frac{2y}{x}$$
$$dy = \frac{2y}{x}dx$$
$$\frac{dy}{2y} = \frac{dx}{x}$$
$$\int \frac{dy}{2y} = \int \frac{dx}{x}$$
$$\frac{1}{2}\log y = \log x + c$$
$$\frac{1}{2}\log y - \log x - c = 0$$

(× by 2)

$$\log y - 2\log x - 2c = 0$$
$$\log y - \log x^2 - c = 0$$
$$\log\left(\frac{y}{x^2}\right) - c = 0$$
$$\left(\frac{y}{x^2} - c\right) = 0$$

case (ii)

$$P = \frac{-y-5y}{2x}$$
$$= \frac{-6y}{2x}$$
$$= \frac{-3y}{x}$$

when $P = -\frac{3y}{x}$

$$\frac{dy}{dx} = \frac{-3y}{x}$$
$$dy = \frac{-3y}{x}dx$$
$$\frac{dy}{3y} = -\frac{dx}{x}$$
$$\int \frac{dy}{3y} = -\int \frac{dx}{x}$$
$$\frac{1}{3}\log y = -\log x + c$$
$$\frac{1}{3}\log y + \log x - c = 0$$

(× by 3)

$$\log y + 3\log x - 3c = 0$$
$$\log y + \log x^3 - 3c = 0$$
$$\log x^3 y - c = 0$$
$$x^3 y - c = 0$$

The combined soln is

$(y - cx^2)(x^3y - c) = 0$ is the req soln.

Exer/12/66P

solve $y - 2px = x^2p^4$ (eqns solvable for y)

soln:

The gn diff eqn is $y - 2px = x^2p^4$

which is of $y = f(x,p)$ $\quad y = 2px + x^2p^4 \longrightarrow (1)$

diff (1) w.r.to x

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4x^2p^3\frac{dp}{dx}$$

$$p = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4x^2p^3\frac{dp}{dx}$$

$$p + 2xp^4 + 2x(1+2xp^3)\frac{dp}{dx} = 0$$

$$p\underbrace{(1+2xp^3)}_{\text{same}} + (1+2xp^3)\,2x\frac{dp}{dx} = 0$$

$$(1+2xp^3) + \left(p + 2x\cdot\frac{dp}{dx}\right) = 0$$

$$1 + 2xp^3 = 0 \quad ; \quad p + 2x\cdot\frac{dp}{dx} = 0$$

case (i)

$1 + 2xp^3 = 0$

$2xp^3 = -1$

$p^3 = \frac{-1}{2x} \rightarrow (2)$

sub (2) in (1)

$y = 2px + x^2p^4$

$= px(2 + xp^3)$

$= px\left(2 + x\left(\frac{-1}{2x}\right)\right)$

$= px\left(2 + \left(\frac{-1}{2}\right)\right)$

$= px\left(\frac{3}{2}\right)$

$y = \frac{3}{2}(px)$

(cube by 3) $y^3 = \frac{27}{8}p^3x^3$

case (ii)

$p + 2x\frac{dp}{dx} = 0$

$2x\frac{dp}{dx} = -p$

$2\frac{dp}{p} = -\frac{dx}{x}$

$2\int\frac{dp}{p} = -\int\frac{dx}{x}$

$2\log p = -\log x + \log c$

$\log p^2 + \log x = \log c$

$\log xp^2 = \log c$

$\boxed{xp^2 = c} \longrightarrow (3)$

$y^3 = \frac{27}{8}\left(\frac{-1}{2x}\right)x^3$

$y^3 = \frac{-27}{16}x^2$

$16y^3 = -27x^2$

given $y = 2px + x^2p^4 \longrightarrow$ ①

eliminate p from ① & ③

$y - x^2p^4 = 2px$

Taking square

$(y - x^2p^4)^2 = 4p^2x^2$

$\boxed{p^2 = c}$

$(y - c^2)^2 = 4x(p^2x)$

$(y-c)^2 = 4xc$ is the general soln of ①

9/3/2020 CLAIRAUT'S FORM :-

the equation of the form

$y = px + f(p) \longrightarrow$ ①

is known as clairauts equation.

the solution of ① is $y = cx + f(c) \rightarrow$ ②

XIII. 9. Exer: $e^{3x}(p-1) + p^3e^{2y} = 0$. solve it.

soln:- gn diff eqn $e^{3x}(p-1) + p^3e^{2y} = 0 \longrightarrow$ ①

Put $x = e^x$, $y = e^y$

$dx = e^x dx$ $\quad dy = \frac{dy}{e^y}$ ey

$dx = \frac{dx}{e^x}$

$dy = dy/ey$

$\frac{dy}{dx} = \frac{dy}{e^y} \cdot \frac{e^x}{dx}$

$\frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx}$

$P = \frac{x}{y} p$

$e^{3x}(p-1) + p^3 e^{2y} = 0 \rightarrow$ ① becomes $x^3\left(\frac{x}{y} p - 1\right) + \frac{x^3}{y^3} p^3 y^2 = 0$

$$x^3\left(\frac{xp-y}{y}\right) + \frac{x^3 p^3}{y} = 0$$

$$x^3(xp-y) + x^3 p^3 = 0$$

$$x^3(xp - y + p^3) = 0$$

$$y = px + p^3$$

which of clairauts equation.

to get solution, we replace P by c.

$$y = cx + c^3$$

$e^y = ce^x + c^3$ is the gen. soln.

$cx = p$

$P = cx$

eliminating P, the soln is

$$2cy = c^2x^2 + 1.$$

Ex : 2 / 62 P

4. solve $x = y^2 + \log p$.

soln :-

gn diff eqn $x = y^2 + \log p$

which is of the form, $x = f(y, p)$

diff y w.r. to y

$$\frac{1}{p} = 2y + \frac{1}{p}\frac{dp}{dy}$$

(x by p)

$$\frac{dp}{dy} + 2py = 1$$

this is linear eqn in P & soln is

$$y e^{\int P dx} + \int Q e^{\int P dx} + c$$

$$P = 2y$$

$$I.F = e^{\int P dy}$$

$$= e^{\int 2y dy}$$

$$I.F = e^{y^2}$$

$P \cdot e^{y^2} = \int e^{y^2} dy + c$ is the Req. soln.

Ex : 1 / 63 P

solve $y = (x-a)P - P^2$

soln :-

gn diff eqn $y = (x-a)P - P^2 \longrightarrow (1)$

This eqn is of the form $y = Px + f(P)$

which is Clairauts form

Hence the soln of (1) is,

$$y = cx + f(c).$$

$$y = (x-a)c - c^2$$

63 P / Ex : 2.

solve $y = 2Px + y^2 P^3$

soln :-

gn diff eqn is $y = 2Px + y^2 P^3$

Put $X = 2x$ & $Y = y^2$

$dX = 2dx$ $\quad dY = 2y dy$

$\frac{dY}{dX} = \frac{2y\,dy}{2dx}$

$$\frac{dy}{dx} = \frac{2dx}{2y\,dy}$$

the eqn transforms into

$$y = xp + p^3$$

$$P = \frac{dy}{dx} - P_y$$

This is Clairauts eqn,

$$y = cx + c^3$$

The soln is $y^2 = 2xc + c^3$.

$$= \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2} = \frac{-xy \pm \sqrt{25x^2y^2}}{2x^2}$$

$$= \frac{-xy \pm 5xy}{2x^2}$$ (Take x as common & cancel) $\frac{x(y \pm 5y)}{2x^2}$

$$P = \frac{-y \pm 5y}{2x}$$

$$P = \frac{-y+5y}{2x} = \frac{4y}{2x}$$

$$P = \frac{2y}{x}$$

$$P = \frac{-y-5y}{2x} = \frac{-6y}{2x}$$

$$P = \frac{-3y}{x}$$

case (i)

when $P = \frac{2y}{x}$

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$dy = \frac{2y}{x}dx$$

$$\frac{dy}{2y} = \frac{dx}{x}$$

$$\int \frac{dy}{2y} = \int \frac{dx}{x}$$

$$\frac{1}{2}\log y = \log x + c$$

$$\frac{1}{2}\log y - \log x - c = 0$$

(mult by 2)

(logm - logn)

$$\log y - 2\log x - 2c = 0$$

$\downarrow$ constant ($\log x^2$)

$$\log\left(\frac{y}{x^2}\right) - c = 0$$

$$\left(\frac{y}{x^2} - c\right) = 0$$

case (ii)

when $P = \frac{-3y}{x}$

$$\frac{dy}{dx} = \frac{-3y}{x}$$

$$\frac{dy}{3y} = \frac{-dx}{x}$$

$$\int \frac{dy}{3y} = -\int \frac{dx}{x}$$

$$\frac{1}{3}\log y = -\log x + c$$

$$\frac{1}{3}\log y + \log x - c = 0$$

($\times$ by 3)

$$\log y + 3\log x - 3c = 0$$

$$\log y + \log x^3 - 3c = 0$$

$$\log x^3y - c = 0$$

$$x^3y - c = 0$$

Ex : XIII / 66P

1. solve $y = Px + \frac{aP}{(1+P^2)^{1/2}}$

sol^n :-

given that $y = Px + \frac{aP}{(1+P^2)^{1/2}} \longrightarrow (1)$

$$\frac{dy}{dx} = P + x\cdot\frac{dP}{dx} + a\left[\frac{\frac{dP}{dx}(1+P^2)^{1/2} - P\,\frac{1}{2}(1+P^2)^{-1/2}\,2P\frac{dP}{dx}}{1+P^2}\right]$$

$$P = P + x\cdot\frac{dP}{dx} + \frac{a}{1+P^2}\frac{dP}{dx}\left((1+P^2)^{\frac{1}{2}} - P^2\frac{1}{(1+P^2)^{1/2}}\right)$$

$$0 = x\cdot\frac{dP}{dx} + \frac{a}{1+P^2}\frac{dP}{dx}\left[\frac{1+P^2-P^2}{(1+P^2)^{\frac{1}{2}}}\right]$$

$$\frac{dP}{dx}\left(x + \frac{a}{(1+P^2)^{3/2}}\right) = 0$$

$$\frac{dP}{dx} = 0 \quad ; \quad x + \frac{a}{(1+P^2)^{3/2}} = 0$$

case (i)

$\frac{dP}{dx} = 0$

$dP = 0$

$\int dP = 0$

$P = c$

sub $P = c$ in (1),

$y = cx + \frac{ac}{(1+c^2)^{1/2}} \longrightarrow (1)$

is gn. sol^n

case (ii)

$x + \frac{a}{(1+P^2)^{3/2}} = 0$

sub $P = c$,

$x + \frac{a}{(1+c^2)^{3/2}} = 0 \longrightarrow (2)$

sub (2) in (1),

$y = c\left(-\frac{a}{(1+c^2)^{3/2}}\right) + \frac{ac}{(1+c^2)^{1/2}}$

$= \frac{ac}{(1+c^2)^{1/2}}\left[-\frac{1}{1+c^2} + 1\right]$

$= \frac{ac}{(1+c^2)^{1/2}}\left[\frac{-1+1+c^2}{1+c^2}\right]$

$y = \frac{ac^3}{(1+c^2)^{3/2}}$

$$y^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}c^2}{(1+c^2)} \longrightarrow ④$$

$$x = -\frac{a}{(1+c^2)^{3/2}}$$

Taking Power $2/3$

$$x^{\frac{2}{3}} = \frac{+a^{\frac{2}{3}}}{(1+c^2)} \longrightarrow ⑤$$

Adding ⑤ & ④

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}}{1+c^2} + \frac{a^{2/3}c^2}{1+c^2}$$

$$= \frac{a^{\frac{2}{3}}}{1+c^2}\left[1+c^2\right]$$

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is singular soln.

(7)

8. XIII. Solve $y = px + \frac{a}{p}$.

Extra

soln :- gn diff eqn, $y = px + \frac{a}{p}$

this is of clairuts form $y = px + f(p)$ ①

Diff ① w.r. to x.

$$\frac{dy}{dx} = p + x\cdot\frac{dp}{dx} + a\left(-\frac{1}{p^2}\right)\left(\frac{dp}{dx}\right)$$

$$\frac{dy}{dx} = p + x\cdot\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

$$p = p + x\cdot\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

Take com $\frac{dp}{dx}$

$$\frac{dp}{dx}\left(x - \frac{a}{p^2}\right) = 0$$

$$\frac{dp}{dx} = 0, \quad x - \frac{a}{p^2} = 0$$

case(i)

$\frac{dP}{dx} = 0$

$dP = 0$

$\int dP = 0$

$P = c$

sub $P = c$ in ①,

$y = cx + \frac{a}{c}$ is the general soln $\longrightarrow$ ②.

case(ii)

$x - \frac{a}{p^2} = 0$

sub $P = c$ in the above

$x - \frac{a}{c^2} = 0 \qquad x = \frac{a}{c^2} \longrightarrow$ ④

To eliminate c, we get singular form ② & ④

from ④,

$c^2 = \frac{a}{x}$

$c = \sqrt{\frac{a}{x}}$

sub the value of c in ②

$y = \sqrt{\frac{a}{x}}(x) + \frac{a}{\sqrt{\frac{a}{x}}}$

$y = \frac{\frac{a}{x}(x) + a}{\sqrt{\frac{a}{x}}}$

$\sqrt{\frac{a}{x}}\, y = 2a$

(by squaring)

$\frac{y^2 a}{x} = 4a^2$

$y^2 = 4a^2 \frac{x}{a}$

$y^2 = 4ax$ is sing soln of ①

19. solve $(Px - y)(Py + x) = 2P$

XIII. soln :-

gn that $(Px - y)(Py + x) = 2P \longrightarrow$ ①

Put $x^2 = X$; $y^2 = Y$

$2x\,dx = dX$ $\qquad 2y\,dy = dY$

$\frac{dy}{dx} = \frac{2y\,dy}{2x \cdot dx}$

$p = \frac{y}{x} P$

$p = \frac{x}{y} P$

eqn ① becomes,

$$\left(\frac{x}{y}Px - y\right)\left(\frac{x}{y}Py + x\right) = 2\frac{x}{y}P$$

$$\left(\frac{x^2P - y^2}{y}\right)\left(xP + x\right) = 2\frac{x}{y}P$$

$$\frac{1}{y}\left(x^2P - y^2\right)x\left(P+1\right) = \frac{2x}{y}P$$

$$\left(x^2P - y^2\right)\left(P+1\right) = 2P$$

$$\left(xP - y\right)\left(P+1\right) = 2P$$

$$xP - y = \frac{2P}{P+1}$$

$$-y = \frac{2P}{P+1} - xP$$

$$y = xP - \frac{2P}{P+1} \longrightarrow ②$$

this is of Clairaut's eqn, $y = Px + f(P)$

to get, gen soln, we replace P by c

$$\therefore y = xc - \frac{2c}{c+1}$$

$$y^2 = x^2c - \frac{2c}{c+1}$$ is gen soln.

H.W

29. XIII. solve :- $(Px - y)(Py + x) = a^2P$ (Put $X = x^2$; $Y = y^2$)

soln :-

G.T. $(Px - y)(Py + x) = a^2P \longrightarrow ①$

Put $X = x^2 \quad Y = y^2$

$$dX = 2x\,dx \qquad dY = 2y\,dy$$

$$\frac{dY}{dX} = \frac{2y\,dy}{2x\,dx}$$

$$P = \frac{y}{x}p \Rightarrow p = \frac{x}{y}P$$

eqn ① becomes

$$\left(x\frac{x}{y}p-y\right)\left(y\frac{x}{y}p+x\right)=a^2\frac{x}{y}p$$

$$\left(\frac{x^2}{y}p-y\right)\left(xp+x\right)=a^2\frac{x}{y}p$$

$$\frac{x}{y}\left(\frac{x^2p-y^2}{y}\right)(p+1)=a^2\frac{x}{y}p$$

$$\left(x^2p-y^2\right)(p+1)=a^2p$$

$x^2=x$, $y^2=y$

$$(xp-y)(p+1)=a^2p$$

$$xp-y=\frac{a^2p}{p+1}$$

$$y=xp=\frac{a^2p}{p+1}$$

which is of clairits eqn,

$$y=p+f(p)$$

we replace, p by c.

$$y=xc-\frac{a^2c}{c+1}$$

$$y^2=-x^2c-\frac{a^2c}{c+1}$$

$$y^2=c\left(x^2-\frac{a^2}{c+1}\right)$$

which is a req. gen form.

28. XIII.

$$x^2(y-px)=p^2y$$

soln :-

G.T. $x^2(y-px)=p^2y \longrightarrow$ ①

Put $\cancel{x} = x^2$, $Y = y^2$

$dX = 2x\,dx$ $\quad dY = 2y\,dy$

$$\frac{dY}{dX} = \frac{y}{x}\frac{dy}{dx}$$

$$P = \frac{y}{x} p$$

$$p = \frac{x}{y} P$$

eqn ① becomes,

$$\cancel{X}\left(y - \frac{x^2 P}{y}\right) = \frac{x^2}{y^2} P^2 y$$

$$x\left(\frac{y^2 - x^2 P}{y}\right) = \frac{x^2}{y^2} P^2$$

$$x\left(y^2 - xP\right) = x\, P^2$$

$$y^2 = xP + P^2$$

$$\cancel{y} = xP + P^2$$

which is of clairuts form.

we Replace P by C,

$$y = xc + c^2$$

$$y^2 = x^2 c + c^2$$

$$y^2 = c(x^2 + c)$$ which is gen soln

7. XIII solve $y^2 = (1+p^2)$

soln:-

$$y^2 = (1+p^2) \longrightarrow ①$$

$$p^2 = y^2 - 1$$

$$p = \sqrt{y^2 - 1}$$

$$\frac{dy}{dx} = \sqrt{y^2 - 1}$$

$$\int \frac{dy}{dx} = \int dx$$

$\log\left[y+\sqrt{y^2+1}\right] = x+c$ is soln

CHAPTER : 5

LINEAR EQN WITH CONSTANT COEFFICIENTS

The general linear differential eqn of the n^{th} order with constant coeffients is of the form

$$\frac{d^n y}{dx} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \frac{a_2\, d^{n-2}y}{dx^{n-2}} + \dots + \dots$$

$$+ a_{n-1}\frac{dy}{dx} + a_n y = x \quad \rightarrow (1)$$

where $a_1, a_2 \dots a_n$ are constant & x is either a fn of x or constant alone.

The complete soln of (1) is,

$$y = C.F + P.I$$

CF - complimentary function &

PI - Particular integral.

THE OPERATOR D :-

Let D denote a operator by $\frac{d}{dx}$

D^2 for $\frac{d^2}{dx^2}$; D^3 for $\frac{d^3}{dx^3}$ etc.

The system D is called a differential operator (or) an operator.

$\therefore$ eqn (1) becomes,

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_{n-1} Dy + a_n y = X$$

$(D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n) y = x$

$f(D) y = x$

where $f(D) = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$

(ie) $f(D)$ is a polynomial in D.

AUXILLARY EQUATION (AE)

The eqn obtained by equating to zero. the coefficient of y is called Auxilary eqn.

RULES FOR FINDING THE COMPLEMENTARY FUNCTION

* CASE (i)

By the roots of auxilary eqn (A.E) are real & distinct

the gen soln of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

PROBLEM :-

11/P/ EG-1

solve $(D^2 - 5D + 4) y = 0$

soln

To find CF

The AE is $m^2 - 5m + 4 = 0$

$(m-4)(m-1) = 0$

$m = 4 \quad m = 1$

$m = 1, 4$

The roots 1, 4 are real & distinct

the gen soln $y = c_1 e^{x} + c_2 e^{4x}$

C_1 & C_2 are arbitary constant

<u>CASE (ii)</u>

two Roots of AE are equal (ie) $m_1 = m_2$

the gen soln of (1) is

$$y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

<u>EG :- 2 / #1</u>

solve $(D^3 - 3D^2 + 4) y = 0$

<u>soln :-</u>

to find C.F

the AE is $m^3 - 3m^2 + 4 = 0$

-1	1	-3	0	4
	↓	-1	4	-4
	1	-4	4	0

$$m^3 - 3m^2 + 4 = (m+1)(m^2 - 4m + 4)$$

$m = -1$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$m = -1, 2, 2$$

the Roots $-1, 2, 2$ are Real & two of the Roots are equal.

The soln is

$$y = c_1 e^{-x} + (c_2 x + c_3) e^{2x}$$

CASE (iii)

two roots of the AC are imaginary the gen soln of ① is

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}.$$

H.W.

1. solve $(D^2 - 8D + 9)y = 0$.

soln :-

$$\begin{array}{c|c} 4 & 28 \\ \hline 2 & 4 \\ \hline 2 & 2 \\ \hline & 1 \end{array}$$

$2\sqrt{7}$

to find CF

the AE is $m^2 - 8m + 9 = 0$

$a = 1 \quad b = -8 \quad c = 9$

$$m = \frac{8 \pm \sqrt{64-36}}{2} = \frac{8 \pm \sqrt{28}}{2}$$

$$= \frac{8 \pm 2\sqrt{7}}{2} = \frac{2(4 \pm \sqrt{7})}{2}$$

$$m = 4 \pm \sqrt{7} \qquad = 4 - \sqrt{7}$$

Real & distinct

soln is $y = c_1 e^{4+\sqrt{7}x} + c_2 e^{4-\sqrt{7}x}$

2) solve $(D^2 - 5D + 6)y = 0$

soln :-

To find C.F

AE is $m^2 - 5m + 6 = 0$

$$m - 2m - 5m + 6 = 0$$

$$m(m-2) - 5(m-2) = 0$$

$$(m-3)(m-2) = 0$$

$m-3=0 \qquad m-2=0$

$m=3 \qquad m=2$

Real & distinct

soln is $y = c_1 e^{2x} + c_2 e^{3x}$

c_1 & c_2 are arbitray constant

3. $(D^2+2D+1)y = 0$

To find CF

AE is $m^2+2m+1=0$

$$m^2+m+m+1=0$$

$$m(m+1)+(m+1)=0$$

$$m=-1,-1$$

Real & equal

gen soln $y = (c_1 x + c_2)e^{-x}$

4. $(D^2-3D+2)y=0$

soln

AE is $m^2-3m+2=0$

$$m^2-2m-m+2=0$$

$$(m-1)(m-2)=0$$

$$m=1 \qquad m=2$$

Roots are 1, 2 & real & Distict

$$y = c_1 e^{x} + c_2 e^{2x}$$

5) $(D^2+5D+6)y=0$

<u>soln</u> :-

AE is $m^2+5m+6=0$

$$\begin{array}{c|c} \multicolumn{2}{c}{6} \\ \hline 2 & 3 \\ m & m^2 \end{array}$$

$$m^2+2m+3m+6=0$$

$$m(m+2)+3(m+2)=0$$

$$(m+3)(m+2)=0$$

$$m=-3 \quad m=-2$$

$-2, \& -3$ are Real & Distinct

$$y = c_1 e^{-2x} + c_2 e^{-3x}.$$

6) $(D^2-6D+13)y=0$

AE is $m^2-6m+13=0$

$$a=1 \quad b=-6 \quad c=13$$

$$= \frac{+6 \pm \sqrt{36-52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm 4i}{2} \quad \frac{2}{2}(3 \pm 2i)$$

$$= 3+2i, \ 3-2i$$

$$y = c_1 e^{(3+2i)x} + c_2 e^{(3-2i)x}.$$

7). $(3D^2+D-14)y=0$

AE is $3m^2+m-14=0$

$$3m^2-6m+7m-14=0$$

$$3m(m-2)+7(m-2)=0$$

$(3m+1)(m-2)=0$

$m = \frac{-1}{3} \qquad m = 2$

$y = c_1 e^{-1/3 x} + c_2 e^{2x}$

8. $(D^2+3D+12)y=0$

soln :-

A.F $\rightarrow m^2 - 13m + 12 = 0$

$m^2 - m - 12m + 12 = 0$

$(m-1)(m-12) = 0$

$m = 1, 12.$

$y = c_1 e^x + c_2 e^{12x}$

9. $(D^3 - 12D + 16)y = 0$

soln :

AF is $m^3 - 12m + 16 = 0$

$$\begin{array}{c|cccc} 2 & 1 & 0 & -12 & 16 \\ & \downarrow & 2 & 4 & -16 \\ \hline & 1 & 2 & -8 & \boxed{0} \end{array}$$

$(m-2)(m^2+2m-8) = 0$

$(m-2)(m^2+4m-2m-8) = 0$

$(m-2)\big(m(m+4)-2(m+4)\big) = 0$

$(m-2)(m-2)(m+4) = 0$

$m = 2, 2, -4$

-8

$4 \quad -2$

$m \quad m$

$y = (c_1 x + c_2)e^{2x} + c_3 e^{-4x}$

10/4 CASE (iii) :-

Eg solve $(D^2 - 6D + 13)y = 0$

AE is $m^2 - 6m + 13 = 0$

By solving $m^2 - 6m + 13 = 0$

$m = 3 \pm 2i$

Roots are imaginary in conjugate pairs

$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$

CASE (iv) :-

Two Pairs of imaginary roots r equal.

The soln is

$$y = e^{\alpha x}\left[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x\right]$$

$$+ c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$$

EXAMPLE :- 3 / 7IP

solve $(D^4 - 4D^3 + 8D^2 - 8D + 4)y = 0$

soln :-

AE is $m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$

$m^4 + 4m^2 + 4 - 4m^3 - 8m + 4m^2 = 0$

$m^4 + 4m^2 + 4$

$(m^2 - 2m + 2)^2 = 0$

$m^2 - 2m + 2 = 0$; $m^2 - 2m + 2 = 0$

$a = 1$ $b = -2$ $c = 2$

$$m = \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= \frac{2(1 \pm i)}{2}$$

$$= 1 \pm i \text{ (twice)}$$

The Roots are imaginary & two pairs of Roots are equal.

$$y = e^x\left[(c_1x + c_2)\cos x + (c_3x + c_4)\sin x\right]$$

Rules for Finding Particular Integral :-

Consider,

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \dots + a_{n-1}\frac{dy}{dx} + a_ny = X$$

ie,

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \dots + a_{n-1}D + a_n)y = X$$

$f(D)y = X$ where $f(D) = D^n + a_1D^{n-1} + \dots + a_{n-1}D + a_n$

$$P.I = \frac{1}{f(D)}X$$

Case (i) :-

when $X = e^{ax}$

$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ Provided $f(a) \neq 0$

If $f(a) = 0$, we have,

$\frac{1}{f(D)}e^{ax} = x \cdot \frac{1}{f'(a)}e^{ax}$ provided $f'(a) \neq 0$

If $f'(a) = 0$ then,

$\frac{1}{f(D)}e^{ax} = x^2 \cdot \frac{1}{f''(a)}e^{ax}$ provided $f''(a) \neq 0$

so on

TAP 11/8/2020

EG : 1

solve $(D^2+5D+6)y = e^x$

soln :-

gn diff eqn is $(D^2+5D+6)y = e^x$

the general form is $y = C.F + P.I$

to find CF

$$(D^2+5D+6)\,y = 0$$

the AE is $m^2+5m+6 = 0$

m common $\quad m^2+2m+3m+6$ take 3

$$m(m+2)+3(m+2) = 0$$

$$m = -3 \quad m = -2$$

the Roots are Real & distinct

$\therefore$ CF is $= c_1 e^{-3x} + c_2 e^{-2x}$

to find P.I $\quad \boxed{D=1}$ $\quad \boxed{e^x \text{ so take } D=1}$

$$PI = \frac{1}{D^2+5D+6}\, e^x$$

$$= \frac{1}{(1+5+6)} e^x$$

$$PI = \frac{1}{12} e^x$$

$\left[\begin{array}{l} f(D) = D^2+5D+6 \\ f(a) = f(1) \\ f(1) = 1+5+6 \\ \quad = 12 \end{array} \right]$

soln is $y = CF + PI$

$$y = c_1 e^{-3x} + c_2 e^{-2x} + \frac{e^x}{12}$$

TAP

EG : 2

solve $(D^2-2mD+m^2)y = e^{mx}$

soln :-

gn $(D^2-2mD+m^2)\,y = e^{mx}$

soln is $y = CF + PI$

To find CF :-

$$D^2-2mD+m^2 = 0$$

the AE is $= k^2-2mk+m^2 = 0$

$$(k-m)(k-m) = 0$$

$$k = m, m$$

the Roots are Real & equal

CF is $(c_1x+c_2)\,e^{mx}$

To find PI

$$PI = \frac{1}{D^2 - 2mD + m^2} e^{mx} \qquad = m^2 - 2m^2 + m^2 = 0$$

$f(D) = D^2 - 2mD + m^2$
$= 0$

If $a = 0$ then $\frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(a)} e^{ax}$ provided $f'(a) \neq 0$

$f'(D) = 2D - 2m$

$f'(m) = 2m - 2m = 0$

If $f(a)' = 0$, then $\frac{1}{f(D)} e^{ax} = x^2 \cdot \frac{1}{f''(a)} e^{ax}$ provide $f''(a) \neq 0$

$f''(D) = 2$

$f''(m) = 2 \neq 0$

$$PI = \frac{1}{D^2 - 2mD + m^2} e^{mx}$$

$$= x^2 \cdot \frac{1}{2} e^{mx} \qquad \left[x^2 \frac{1}{f''(a)} e^{ax}\right]$$

$$= \frac{x^2 \cdot e^{mx}}{2}$$

soln is $y = CF + PI$

$$y = (c_1 x + c_2) e^{mx} + \frac{x^2 e^{mx}}{2}$$

EXER :- XIV / 86P.

(1) solve $(D^2 - 5D + 6)y = e^{4x}$

soln :-

To find CF

The AE is $m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0$

$m = 3 \quad m = 2$

The roots are real and distinct

$CF = c_1 e^{3x} + c_2 e^{2x}$

To find PI :-

$$PI = \frac{1}{D^2-5D+6} e^{4x} \quad \boxed{D=4}$$

$$= \frac{1}{16-20+6} e^{4x}$$

$16-20+6$
$22-20$
$=2$

$$= \frac{1}{2} e^{4x}$$

soln is $y = CF + PI$

$$y = c_1e^{3x} + c_2e^{2x} + \frac{e^{4x}}{2}$$

XIV. ② $(D^2+2D+1)y = 2e^{3x}$

soln :-

To find CF :-

The AE is $m^2+2m+1 = 0$

$$m(m+1)+1(m+1) = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1$$

The CF is $(c_1x + c_2)e^{-1x}$

To find PI :- $\boxed{D=3}$

$$PI = \frac{1}{D^2+2D+1} 2e^{3x}$$

$$= \frac{1}{9+6+1} 2e^{x}$$

$$= \frac{2e^{3x}}{16}$$

$$= \frac{e^{3x}}{8}$$

$$y = (c_1x+c_2)^{-x} + \frac{e^{3x}}{8}$$

XIV ③ solve $(D^2-3D+2)y = e^{3x}$.

soln :-

The AE is $m^2-3m+2=0$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$$CF = c_1e^{x} + c_2e^{2x}$$

$$PI = \frac{1}{D^2-3D+2} e^{3x}$$

$= \frac{1}{9-9+2} e^{3x}$ $\boxed{m=3}$

$= \frac{1}{2} e^{3x}$

$y = c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{2}$

$\overline{XIV}$

4. $(D^2+5D+6)y = e^{x}$

soln

The AE is $m^2+5m+6=0$

$(m+3)(m+2)=0$

$m=-3,\ m=-2$

The CF is $= c_1 e^{-2x} + c_2 e^{-3x}$

$PI = \frac{1}{D^2+5D+6} e^{x}$ $\boxed{D=1}$

$= \frac{1}{1+5+6} e^{x}$

$= \frac{e^{x}}{12}$

$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^{x}}{12}$

$\overline{XIV}$

5. $(D^2-6D+13)y = 5e^{2x}$

soln :-

The AE is $m^2-6m+13=0$

$= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$a=1\ \ b=-6\ \ c=13$

$= \frac{6 \pm \sqrt{36-52}}{2}$

$= \frac{6 \pm \sqrt{-16}}{2}$

$= \frac{6 \pm 4i}{2}$

$= \frac{\cancel{2}(3 \pm 2i)}{\cancel{2}}$

CF is $c_1 e^{3+2i} + c_2 e^{3-2i}$

To find PI $\boxed{D=2}$

$$PI = \frac{1}{D^2-6D+13} \times 5e^{2x}$$

$$= \frac{1}{4-12+13} \times 5e^{2x}$$

$$= \frac{1}{5} \times 5e^{2x}$$

$$y = C_1 e^{3+2i} + C_2 e^{3-2i} + e^{2x}$$

XIV

6) $(3D^2+D-14)y = 12e^{2x}$

soln :-

The AE is $3m^2+m-14=0$

$$3m(m-2)+7(m-2)=0$$

$$3m+7=0 \qquad m-2=0$$

$$m = \frac{-7}{3} \qquad m=2$$

CF is $y = C_1 e^{-\frac{7}{3}x} + C_2 e^{2x}$

To find PI

$$PI = \frac{1}{3D^2+D-14} 13e^{2x} \quad \boxed{D=2}$$

$$= \frac{1}{12+2-14} 13e^{2x}$$

$$= 0$$

If $a=0$ the $\frac{1}{f(0)} e^{ax} = x \cdot \frac{1}{f'(a)} e^{ax}$

$f'(D) = 6D+1$

$f'(m) = 6m+1 \quad \boxed{m=2}$

$= 12+1 = 13$

$$= x \cdot \frac{\cancel{13} e^{2x}}{\cancel{13}} \quad \left[x \cdot \frac{1}{f'(a)} e^{ax}\right]$$

$$= x \cdot e^{2x}$$

$$y = (C_2+x)e^{2x} + C_1 e^{-7/3 x} + xe^{2x}$$

XIV

7. $(D^2-13D+12)y = e^{-2x}+5e^{x}$

soln:-

The AE is $m^2-13m+12=0$

$(m-1)(m-12)=0$

$m=1, 12$

CF $\Rightarrow y = c_1e^{x}+c_2e^{12x}$

PI :- $\frac{1}{D^2-13D+12} \times e^{-2x}$ $D=-2$

$= \frac{1}{4+26+12} e^{-2x}$

$= \frac{e^{-2x}}{42}$

$PI = \frac{1}{D^2-13D+12} \times 5e^{x}$ $\boxed{D=1}$

$= \frac{1}{1-13+12}$

$= \frac{1}{0} \times 5e^{x} = 0$

If $a=0$ $f'(D) = \frac{1}{2D-13} = \frac{1}{2-13} = \frac{5}{-11}e^{x}$

$y = c_1e^{x}+c_2e^{12x}+\frac{e^{-2x}}{42}-\frac{5e^{x}}{11}$

XIV

8. $(D^3-12D+16)y = (e^{x}+e^{-2x})^2$

soln:-

The AE is $m^3-12m+16=0$

$m^3-12m+16=0$

$(m-2)(m^2+2m-8)=0$

$(m-2)(m-2)(m+4)=0$

$m=2, 2, -4$

The CF is $(c_1x+c_2)e^{2x}+c_3e^{-4x}$

$(e^{x}+e^{-2x})^2 = e^{2x}+e^{-4x}+2e^{x}e^{-2x}$

$(a+b)^2$

$$PI = \frac{1}{D^3-12D+16} e^{2x} \quad \boxed{D=2}$$

$$= \frac{1}{8-24+16} e^{2x}$$

$$= \frac{e^{2x}}{0} = 0$$

If $f(a) = 0$

$$f'(D) = 3D^2 - 12 \quad \boxed{D=2}$$

$$= 12 - 12 = 0$$

$$f''(D) = 6D$$

$$= 12$$

$$\underline{D.I} = \frac{e^{2x}}{12}$$

$$PI = \frac{1}{D^3-12D+6} e^{-4x} \quad \boxed{D=-4}$$

$$= \frac{1}{-64+48+6} e^{-4x}$$

$$= \frac{e^{-4x}}{-10}$$

$$PI = 2e^{-x}$$

$$= \frac{1}{D^3-12D+6} 2e^{-x}$$

$$= \frac{1}{-1+12+6} 2e^{-x}$$

$$= \frac{2e^{-x}}{17}$$

$$PI = \frac{e^{2x}}{12} - \frac{e^{-4x}}{10} + \frac{2e^{-x}}{17}$$

$$y = (c_1x+c_2)e^{2x} + c_3e^{-4x} + \frac{x^2e^{2x}}{12} - \frac{xe^{-4x}}{36} + \frac{2e^{-x}}{17}$$

<u>CASE II :-</u>

when x is of the form x^m, m being a +ve integer to evaluate, $\frac{1}{f(D)} x^m$

$$PI = \frac{1}{f(D)} x^m$$

$$= [f(D)]^{-1} x^m$$

PROBLEMS

Ex:- 1 $32/P$.

solve $(D^3 - D^2 - D + 1)y = 1 + x^2$

soln :-

Given eqn is $(D^3 - D^2 - D + 1)y = 1 + x^2$

To find CF :-

The AE is $m^3 - m^2 - m + 1 = 0$

$(m-1)(m^2-1) = 0$

$(m-1)(m-1)(m+1) = 0$

$m = 1, 1, -1$

$m = 1$, twice, $m = -1$

$\therefore$ the Roots are Real & two of them are equal

$CF = c_1 e^{-x} + (c_2 x + c_3) e^x$.

To find PI :-

$$PI = \frac{1}{D^3 - D^2 - D + 1}(1+x^2)$$

$= (D^3 - D^2 - D + 1)^{-1}(1+x^2)$

$= [(D-1)(D^2-1)]^{-1}(1+x^2)$ → by formula

$= [(1-D)(1-D^2)]^{-1}(1+x^2)$

$= (1-D)^{-1}(1-D^2)^{-1}(1+x^2)$

$= (1+D+D^2)(1+D^2)(1+x^2)$

[omitting D^3 & higher Power of

$= (1 + D^2 + D + D^2)(1+x^2)$

$= (1 + D + 2D^2)(1+x^2)$

$= (1+x^2) + D(1+x^2) + 2D^2(1+x^2)$

$= 1 + x^2 + 2x + 2(2)$

$= 1 + x^2 + 2x + 4$

$PI = x^2 + 2x + 5$

$D^3 - D^2 - D + 1$

$D^2(D-1) - 1(D-1)$

$(D-1)(D^2-1)$

Double diff (D^2)

$1+x^2$

$\Rightarrow 2x$

$\Rightarrow 2$

$y = CF + PI$

$y = c_1 e^{-x} + (c_2 x + c_3) e^{x} + x^2 + 2x + 5.$

CASE iii :-

when x is of the form cos ax (or) sin ax where a is constant.

$PI = \frac{1}{\phi(D)^2} \cos ax$ (ϕ - Pie)

$\frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax$ if $\phi(-a^2) \neq 0$

$\frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax$ if $\phi(-a^2) \neq 0$

$\frac{1}{D^2+a^2} \sin ax = \frac{-x \cos ax}{2a}$ if $\phi(-a^2) = 0$

$\frac{1}{D^2+a^2} \cos ax = \frac{x \sin ax}{2a}$ if $\phi(-a^2) = 0$.

PROBLMS :-

XV.

12. solve $(D^2 - 2D - 8)y = 4 \cos 2x$.

soln :-

Gn eqn is $(D^2 - 2D - 8)y = 4 \cos 2x$.

To find CF :-

the AE is $m^2 - 2m - 8 = 0$

$m^2 - 2m - 8 = 0$

$m^2 + 2m - 4m - 8 = 0$

$m(m+2) - 4(m+2) = 0$

$(m-4)(m+2) = 0$

$m = 4, -2$

the CF is $c_1 e^{4x} + c_2 e^{-2x}$

To find PI

$$PI = \frac{1}{D^2-2D-8} \, 4\cos 2x$$

$$= 4 \cdot \frac{1}{-4-2D-8} \cos 2x \qquad [D^2 = -a^2 = -4]$$

$$= 4 \cdot \frac{1}{-2D-12} \cos 2x$$

(Take -2)

$$= \frac{\cancel{4}^{2}}{-\cancel{2}} \frac{1}{D+6} \cos 2x$$

($\times_r \div$ by $D-6$)

$$= -2 \frac{D-6}{(D+6)(D-6)} \cos 2x$$

$$= -2 \frac{D-6}{D^2-36} \cos 2x$$

$$= -2 \frac{D-6}{-4-36} \cos 2x \quad \text{Here } \left[\begin{array}{l} D^2 = -a^2 \\ = -4 \end{array}\right]$$

$$= -\cancel{2} \cdot \frac{D-6}{-\cancel{40}_{20}} \cos 2x$$

$$= \frac{1}{20} D(\cos 2x) - 6\cos 2x$$

$$PI = \frac{1}{20}(-2\sin 2x - 6\cos 2x)$$

$$\frac{-2\sin 2x}{20} - \frac{6\cos 2x}{20}$$

$$PI = \frac{-\sin 2x}{10} - \frac{3\cos 2x}{10}$$

$$y = c_1 e^{-2x} + c_2 e^{4x}.$$

$$\frac{\sin 2x}{10} - \frac{3\cos 2x}{10}$$

9. solve $(D^2+D+1)y = \sin 2x$

pw.

soln :-

to find CF

the AE is $m^2+m+1=0$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$a=1 \quad b=1 \quad c=1$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$= \frac{-1}{2} \pm \frac{\sqrt{3}}{2}$$

the CF is $c_1 e^{-x/2} \cos\left(\frac{\sqrt{3}}{2}x + c_2\right)$

to find PI

$$= \frac{1}{f(D^2)} x^m$$

$$= \frac{1}{D^2+D+1} \sin 2x$$

$$= \frac{1}{4+D+1} \sin 2x$$

$$= \frac{\sin 2x}{D+5}$$

$$= \frac{D-5}{(D+5)(D-5)} \sin 2x$$

$$= \frac{1}{D+5)}$$

$$= \frac{D-5}{D^2-5^2} \sin 2x$$

$$= \frac{D-5}{4-25} \sin 2x$$

$$= -\frac{1}{21} D(\sin 2x) - 5(\sin 2x$$

$$= \frac{1}{21} 2\cos 2x - 5\sin 2x$$

$$PI = \frac{-2\cos 2x + 5\sin 2x}{21}$$

10. XIV solve $(D^2 - 8D + 9)y = 8\cos 5x$

soln :-

To find CF :-

The AE is $m^2 - 8m + 9 = 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1 \quad b = -8 \quad c = 9$

$$\frac{8 \pm \sqrt{64 - 36}}{2}$$

$$= \frac{8 \pm \sqrt{28}}{2}$$

$$= \frac{8 \pm 2\sqrt{7}}{2}$$

$$= \frac{2(4 \pm \sqrt{7})}{2}$$

$m = 4 \pm \sqrt{7}$

the CF is $= c_1 e^{4+\sqrt{7}x} + c_2 e^{4-\sqrt{7}x}$

To find PI.

$$PI = \frac{1}{D^2 - 8D + 9} \; 8\cos 5x$$

$D^2 = -a^2 = -5^2 = -25$

$$= \frac{1}{-25 - 8D + 9} \times 8\cos 5x$$

$$= \frac{1}{-8D - 16} \; 8\cos 5x$$

$$= \frac{1}{-\cancel{8}(D+2)} \; \cancel{8}\cos 5x$$

$$= \frac{-1}{D+2} \cos 5x$$

$$\left[\times \text{ by } (D-2) \right]$$

$$= \frac{-(D-2)}{(D+2)(D-2)} \cos 5x$$

$$= \frac{-(D-2)\cos 5x}{D^2-4}$$

$$= \frac{-(D-2)\cos 5x}{-25-4} \qquad [D^2 = -a^2 = -25]$$

$$= \frac{-(D-2)\cos 5x}{-29}$$

$$= \frac{(D-2)\cos 5x}{29}$$

$$= \frac{1}{29}(D\cos 5x - 2\cos 5x)$$

$$= \frac{1}{29} - 5\sin 5x - 2\cos 5x$$

$$\underline{PI} = \frac{-5\sin 5x}{29} - \frac{2\cos 5x}{29}$$

$$y = e^{4x}\left[c_1 e^{\sqrt{7}x} + c_2 e^{-\sqrt{7}x}\right] - \frac{5\sin x}{29} - \frac{2\cos 5x}{29}$$

XIV

11. solve $(D^2+4)y = \sin 3x + \cos 2x$

<u>soln</u> :-

<u>to find CF</u>

<u>the AE is</u> $\quad m^2+4 = 0$

$$m^2 = -4$$

$$m = \pm 2i$$

$$C.F = [c_1\cos 2x + c_2\sin 2x]$$

<u>to find PI</u>

$$PI = \frac{1}{D^2+4}\sin 3x + \cos 2x$$

$$= \frac{\sin 3x}{D^2+4} + \frac{\cos 2x}{D^2+4}$$

<u>Consider</u> $\frac{\sin 3x}{D^2+4} = \frac{1}{-9+4}\sin 3x \qquad D^2 = -a^2 = -$

$$PI = \frac{-\sin 3x}{5}$$

Consider $PI = \frac{\cos 2x}{D^2+4}$

$\frac{1}{D^2+a^2} \cos ax = \frac{x \sin ax}{2a}$; $+ \; 4(-9)^2 = 0$

$PI = \frac{x \sin 2x}{2(2)}$

$PI = \frac{x \sin 2x}{4}$

$PI = \frac{-\sin 3x}{5} + \frac{x \sin 2x}{4}$

$y = [c_1 \cos 2x + c_2 \sin 3x] - \frac{\sin 3x}{5} + \frac{x \sin 2x}{4}$

15. XIV solve $(D^2-D-2)y = 20\sin 2x + 4e^{3x}$

soln :-

to find CF

the AE is $m^2-m-2=0$

$(m+1)(m-2)=0$

$m=-1,2$

$CF = c_1 e^{-x} + c_2 e^{2x}$

to find PI

$PI = \frac{1}{D^2-D-2} 20\sin 2x + \frac{1}{D^2-D-2} 4e^{3x}$

$PI = \frac{1}{D^2-D-2} 20 \sin 2x$ $\left[D^2 = -a^2 = -2^2 = \right.$

$= \frac{1}{-4-D-2} 20 \sin 2x$

$= \frac{1}{-D-6} 20 \sin 2x$

$= \frac{-20 \sin 2x}{D+6}$

$= \frac{-(D-6) \, 20 \sin 2x}{(D+6)(D-6)}$

$$= \frac{-(D-6)\, 20 \sin 2x}{D^2-36} \qquad [D^2 = -a^2 = -2^2 = -4]$$

$$= -20 \frac{(D-6) \sin 2x}{-4-36}$$

$$= \frac{-20}{-40} (D-6) \sin 2x$$

$$= \frac{1}{2} D \sin 2x - 6 \sin 2x$$

$$= \frac{1}{2}\, 2 \cos 2x - 6 \sin 2x$$

$$= 2 \frac{\cos 2x}{2} - 6 \frac{\sin 2x}{2}$$

Consider, $PI = \frac{1}{D^2-D-2} 4e^{3x}$

$$= \frac{4e^{3x}}{3^2-3-2} \qquad \boxed{D=3}$$

$$= \frac{4e^{3x}}{9-5}$$

$$= \frac{4e^{3x}}{4}$$

$$PI = e^{3x}$$

$$y = c_1 e^{-x} + c_2 e^{2x} + \cos 2x - 3 \sin 2x + e^{3x}$$

16. Solve $(D^2-4D-5)y = e^{2x} + 3\cos 4x$.

XIV

soln :-

To find CF

The AE is $m^2 - 4m - 5 = 0 \cancel{e^{2x} + 3\cos 4x}$

$$(m-5)(m+1) = 0$$

$$m = 5, -1$$

$$CF = c_1 e^{-x} + c_2 e^{5x}$$

To find PI

$$PI = \frac{1}{D^2-4D-5} e^{2x} + \frac{1}{D^2-4D-5} 3\cos 4x \longrightarrow (1)$$

consider

$$PI = \frac{1}{(D^2-4D-5)} e^{2x} \qquad D = 2$$

$$= \frac{1}{2^2-4(2)-5} e^{2x}$$

$$PI = \frac{e^{2x}}{-9}$$

consider,

$$PI = \frac{1}{D^2-4D-5} 3\cos 4x \qquad [D^2 = -a^2 = -4^2 = -16]$$

$$= \frac{1}{-16-4D-5} 3\cos 4x$$

$$= \frac{1}{-4D-21} 3\cos 4x$$

$$= \frac{-3\cos 4x}{4D+21}$$

$$= \frac{-3(4D-21)}{(4D+21)(4D-21)} \cos 4x$$

$$= \frac{-3(4D-21)}{16D^2 - 441} \cos 4x$$

$$= \frac{-3(4D-21)}{16(-16)-441} \cos 4x$$

$$= \frac{+3(4D-21)}{+697} \cos 4x$$

$$= \frac{3}{697}(4D\cos 4x - 21\cos 4x)$$

$$= \frac{3}{697}(-16\sin 4x - 21\cos 4x)$$

① $\Rightarrow$

$$PI = \frac{-e^{2x}}{9} - \frac{3}{697}(16\sin 4x + 21\cos 4x)$$

$$y = c_1e^{-x} + c_2e^{5x} - \frac{e^{2x}}{9} - \frac{3}{697}(16\sin 4x + 21\cos 4x)$$

XIV (17) hw

solve $(D^2+5D+6)y = e^{-2x} + \sin 4x$

sol.n

to find CF

The AE is $m^2+5m+6=0$

$(m+3)(m+2)=0$

$m=-3,-2$

CF is $c_1 e^{-3x} + c_2 e^{-2x}$

to find PI

$$PI = \frac{1}{D^2+5D+6} e^{-2x} + \frac{1}{D^2+5D+6} \sin 4x \longrightarrow (1)$$

$$PI = \frac{1}{D^2+5D+6} e^{-2x}$$

If $f(a)=0$, we have

$$\frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(a)} e^{ax}$$

$f'(D) = 2D+5$

$= 2(-2)+5$

$= -4+5$

$f'(D) = 1 \neq 0$

$$PI = x \cdot e^{-2x} \longrightarrow (2)$$

Consider, $\frac{1}{D^2+5D+6} \sin 4x$.

$$PI = \frac{1}{-16+5D+6} \sin 4x$$

$$= \frac{1}{5D-10} \sin 4x$$

$$= \frac{1}{5(D-2)} \sin 4x$$

$$= \frac{1}{5} \frac{(D+2)}{D^2-4} \sin 4x$$

$$= \frac{1}{5} \frac{D+2}{-20} \sin 4x$$

$= \frac{-1}{100} (D+2) \sin 4x$

$= \frac{-1}{100} D \sin 4x + 2 \sin 4x$

$= \frac{-1}{100} (4 \cos 4x + 2 \sin 4x)$

$= \frac{-4 \cos 4x}{100} - \frac{2 \sin 4x}{100}$

$y = c_1 e^{-3x} + c_2 e^{-2}x + x \cdot e^{-2x} - \frac{4 \cos 4x}{100} - \frac{2 \sin 4x}{100}$

33. solve $(D^2-1)y = 2+5x$

XIV

hw

soln :-

To find CF

$(D^2-1)y = 0$

$m^2-1 = 0$

$m^2 = 1$

$m = \pm 1$

$CF = c_1 e^{x} + c_2 e^{-x}$

$PI = \frac{1}{D^2-1} (2+5x)$

$= (D^2-1)^{-1} (2+5x)$

$= [(D+1)(D-1)]^{-1} (2+5x)$

$= (D+1)^{-1} (D-1)^{-1} (2+5x)$

$= (1-D)^{-1} (1+D)^{-1} (2+5x)$

$= (1+D)(1-D)(2+5x)$

$= (1-D+D-D^2)(2+5x)$

$= 1(2+5x)$

$PI = 2+5x$

$y = c_1 e^{x} + c_2 e^{-x} + 2 + 5x$

CASE, IV:

When X is of the form xV where V is some function of x

$$PI = \frac{1}{f(D)} xV$$

$$= \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} V.$$

PROBLEM

EG :- 4

Solve $(D^2+4)y = x \sin x$

Soln :-

To find CF

The AE is $m^2 + 4 = 0$

$$m^2 = -4$$

$$m = \pm 2i$$

The Roots are imaginary.

$$CF = c_1 \cos 2x + c_2 \sin 2x.$$

To find PI :-

$$PI = \frac{1}{D^2+4} x \sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{D^2+4} \sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{-1+4} \sin x$$

sin value ① so $D^2 = -1$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{3} \sin x$$

$$= \frac{x}{3} \sin x - \frac{2}{3} \frac{D}{D^2+4} \sin x$$

$$= \frac{x}{3} \sin x - \frac{2D}{3} \frac{1}{D^2+4} \sin x$$

$= \frac{x}{3}\sin x - \frac{2D}{3}\frac{1}{-1+4}\sin x$

$= \frac{x}{3}\sin x - \frac{2}{3}D\frac{1}{3}\sin x$

$= \frac{x}{3}\sin x - \frac{2}{9}D\sin x$

$= \frac{x}{3}\sin x - \frac{2}{9}\cos x$

$y = CF + PI$

$y = c_1\cos 2x + c_2\sin 2x + \frac{x\sin x}{3} - \frac{2\cos x}{9}$

CASE V :-

when x is of the form $e^{ax}V$ where V is any func of x,

$$PI = \frac{1}{f(D)}e^{ax}V$$

$$PI = e^{ax}\frac{1}{f(D+a)}V.$$

PROBLEM

EG :-

solve $(D^3 - 2D + 4)y = e^x\cos x$

soln :-

to find CF

the AE is $m^3 - 2m + 4 = 0$

$(m+2)(m^2 - 2m + 2) = 0$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $a = 1$ $b = -2$ $c = 2$

$a = 1$ $b = -2$ $c = 2$

$\frac{-2 \pm \sqrt{4-8}}{2}$

$= \frac{2 \pm \sqrt{-4}}{2} \Rightarrow \frac{2 \pm 2i}{2}$

$$= \frac{2(1\pm i)}{2}$$

$m = -2$

$m = 1 \pm i$

the roots are real & imaginary

$$CF = c_1 e^{-2x} + e^{x}(c_2 \cos x + c_3 \sin x)$$

<u>to find P.I</u>

$$PI = \frac{1}{D^3 - 2D + 4} e^{x} \cos x$$

$$= e^{x} \frac{1}{(D+1)^3 - 2(D+1) + 4} \cos x \quad a = 1$$

$$= e^{x} \frac{1}{D^3 + 3D^2 + 3D + 1 - 2D - 2 + 4} \cos x$$

$$= e^{x} \frac{1}{D^3 + 3D^2 + D + 3} \cos x$$

$$= e^{x} \frac{1}{D^2(D+3) + 1(D+3)} \cos x$$

$$= e^{x} \frac{1}{(D^2+1)(D+3)} \cos x$$

$$= e^{x} \frac{1}{D+3} \frac{1}{D^2+1} \cos x$$

$$\frac{1}{D^2+a^2} x \cos ax = \frac{x \sin}{2}$$

$$= e^{x} \frac{1}{D+3} \frac{x \sin x}{2}$$

$$= \frac{e^{x}}{2} \cdot \frac{1}{D+3} x \sin x$$

$$= \frac{e^{x}}{2} \left[x - \frac{1}{D+3}\right] \frac{1}{D+3} \sin x \quad \text{(conjugate)}$$

$$= \frac{e^{x}}{2} \left[x - \frac{1}{D+3}\right] \frac{D-3}{(D+3)(D-3)} \sin x$$

$$= \frac{e^x}{2}\left[x - \frac{1}{D+3}\right]\frac{D-3}{D^2-9}\sin x \qquad D^2 = -a^2$$

$$= \frac{e^x}{2}\left[x - \frac{1}{D+3}\right]\frac{D-3}{-1-9}\sin x$$

$$= \frac{e^x}{2}\left[x - \frac{1}{D+3}\right]\frac{D-3}{-10}\sin x$$

$$= \frac{-e^x}{20}\left[x - \frac{1}{D+3}\right]\underbrace{D\sin x}_{\text{diff}} - 3\sin x$$

$$= \frac{-e^x}{20}\left[x - \frac{1}{D+3}\right]\cos x - 3\sin x$$

$$= \frac{-e^x}{20}\left(x\cos x - 3x\sin x - \frac{\cos x}{D+3} + \frac{3\sin x}{D+3}\right)$$

$\times / \div\ D-3$ $\qquad$ $\times / \div\ D-3$

$$= \frac{-e^x}{20}\left(x\cos x - 3x\sin x - \frac{D-3}{D^2-9}\cos x + \frac{3(D-3)}{D^2-9}\sin x\right)$$

$$= \frac{-e^x}{20}\left(x\cos x - 3x\sin x + \frac{81}{10}D\cos x - \frac{3\cos x}{10} - \frac{3}{10}D\sin x + \frac{9}{10}\sin x\right]$$

$$= \frac{-e^x}{20}\left[x\cos x - 3x\sin x - \frac{\sin x}{10} - \frac{3\cos x}{10} - \frac{3\cos x}{10} + 9\frac{\sin x}{10}\right]$$

$$= \frac{-e^x}{20}\left[x\cos x - 3x\sin x + \frac{8\sin x}{10} - \frac{6}{10}\cos x\right]$$

$$y = c_1 e^{-2x} + e^x(c_2\cos x + c_3\sin x) + -\frac{e^x}{20}\left(x\cos x - 3x\sin x + 8\frac{\sin x}{10} - \frac{6}{10}\cos x\right)$$

Eg : 3

show that the soln of the differential equation :

10 mk $\frac{d^2y}{dt^2} + 4y = A \sin pt$ which is such that $y = 0$ &

(*) repeated $\frac{dy}{dt} = 0$ when $t = 0$ is $y = \frac{A(\sin pt - \frac{1}{2} P \sin 2t)}{4 - p^2}$ if $P \neq 2$

if $P = 2$, s.T. $y = \frac{A(\sin 2t - 2t \cos 2t)}{8}$

soln :-

$$g.T. \quad \frac{d^2y}{dt^2} + 4y = A \sin pt \qquad \boxed{\frac{d^2y}{dt^2} = D^2}$$

$$(D^2+4)y = A \sin pt \longrightarrow (1)$$

to find CF

the AE is $m^2 + 4 = 0$

$$m^2 = -4$$

$$m = \pm 2i$$

$$m = \pm 2i$$

$$CF = c_1 \cos 2t + c_2 \sin 2t$$

to find PI

$$PI = \frac{1}{D^2+4} A \sin pt$$

$$= A \cdot \frac{1}{D^2+4} \sin pt \qquad D^2 = -a^2 = -p^2$$

$$= A \cdot \frac{1}{-p^2+4} \sin pt$$

$$PI = \frac{A \sin pt}{-p^2+4}$$

$$y = CF + PI$$

$$y = c_1 \cos 2t + c_2 \sin 2t + \frac{A \sin pt}{4 - p^2} \longrightarrow (2)$$

when $t=0$, $y=0$ & $\frac{dy}{dt}=0$

when $t=0$, $y=0$

$$\boxed{0=c_1}$$

when $t=0$, $\frac{dy}{dt}=0$

diff ② w.r.t 't', we get,

$y = c_1\cos 2t + c_2 \sin 2t + \frac{A\sin pt}{4-p^2}$ → ②

$$\frac{dy}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{Ap\cos pt}{4-p^2}$$

$c_1=0$ $\cos 0 = 1$

$$0 = 2c_2 + \frac{Ap}{4-p^2}$$

$$2c_2 = \frac{-Ap}{4-p^2}$$

$$\boxed{c_2 = \frac{-Ap}{8-2p^2}} \rightarrow 2(4-p^2)$$

sub the values c_1 & c_2 in ②,

$$y = 0 - \frac{Ap\sin 2t}{8-2p^2} + \frac{A\sin pt}{4-p^2}$$

$$= \frac{-Ap\sin 2t + 2A\sin pt}{2(4-p^2)}$$

$$y = \frac{A}{2(4-p^2)}\left(2\sin pt - p\sin 2t\right)$$

$$y = \frac{A}{4-p^2}\left(\sin pt - \frac{p}{2}\sin 2t\right)$$

when $p=2$,

$$PI = \frac{1}{D^2+4} A\sin 2t$$

$$= A \cdot \frac{1}{D^2+4}\sin 2t$$

$$= A\left(-\frac{t\cos 2t}{4}\right)$$

$$= \frac{A}{4} - t\cos 2t$$

$$y = c_1\cos 2t + c_2\sin 2t - \frac{At\cos 2t}{4} \rightarrow (3)$$

when $t = 0$, $y = 0$

$$c_1 = 0$$

diff (3) w.r.t "t"

$$\frac{dy}{dt} = -2c_1\sin 2t + 2c_2\cos 2t - \frac{A}{4}(\cos 2t - 2t\sin 2t)$$

$$\frac{dy}{dt} = 0, \quad c = 0, \quad t = 0$$

$$0 = 2c_2 - \frac{A}{4}$$

$$2c_2 = \frac{A}{4}$$

$$c_2 = \frac{A}{8}$$

(3) $\Rightarrow$

$$y = \frac{A\sin 2t}{8} - \frac{At\cos 2t}{4}$$

$$= \frac{A(\sin 2t)}{8} - \frac{At\cos 2t}{4} \times 2$$

$$= \frac{A\sin 2t - 2At\cos 2t}{8}$$

$$= \frac{A(\sin 2t - 2t\cos 2t)}{8}$$

$$= A\left(-\frac{t\cos 2t}{4}\right)$$

$$= \frac{A}{4} \cdot -t\cos 2t$$

$$y = c_1\cos 2t + c_2\sin 2t - \frac{At\cos 2t}{4} \rightarrow (3)$$

when $t=0$, $y=0$

$$c_1 = 0$$

diff (3) w.r.t "t"

$$\frac{dy}{dt} = -2c_1\sin 2t + 2c_2\cos 2t - \frac{A}{4}(\cos 2t - 2t\sin 2t)$$

$\frac{dy}{dt} = 0$, $c=0$, $t=0$

$$0 = 2c_2 - \frac{A}{4}$$

$$2c_2 = \frac{A}{4}$$

$$c_2 = \frac{A}{8}$$

(3) $\Rightarrow$

$$y = \frac{A\sin 2t}{8} - \frac{At\cos 2t}{4}$$

$$= \frac{A(\sin 2t)}{8} - \frac{At\cos 2t}{4} \times 2$$

$$= \frac{A\sin 2t - 2At\cos 2t}{8}$$

$$= \frac{A(\sin 2t - 2t\cos 2t)}{8}$$

Eg: A

solve $(D^2-4D+3)y = \sin 3x \cos 2x$

soln :-

to find CF

the AE is $m^2-4m+3=0$

$m^2-m-3m+3=0$

$m(m-1)-3(m-1)=0$

$(m-1)(m-3)=0$

$m=1,3$

$(m-3)(m-1)$

$m=1,3$

$CF = c_1e^x + c_2e^{3x}$

to find PI

$\sin A \cos B = \frac{\sin(A+B)+\sin(A-B)}{2}$

$PI = \frac{1}{D^2-4D+3} \sin 3x \cos 2x$

$= \frac{\sin(3x+2x)+\sin(3x-2x)}{2}$

$= \frac{1}{D^2-4D+3} \frac{\sin 5x + \sin x}{2}$

$= \frac{1}{D^2-4D+3}\frac{\sin 5x}{2} + \frac{1}{D^2-4D+3}\frac{\sin x}{2}$

$= \frac{1}{2}\frac{\sin 5x}{-25-4D+3} + \frac{1}{2}\frac{\sin x}{-1-4D+3}$

$= \frac{1}{2}\left[\frac{\sin 5x}{-4D-22} + \frac{\sin x}{-4D+2}\right]$

$= \frac{1}{4}\left[\frac{\sin 5x}{-2D-11} + \frac{\sin x}{1-2D}\right]$

$= -\frac{1}{4}\left[\frac{2D-11 \sin 5x}{(2D-11)(2D+11)} + \frac{1+2D}{(1-2D)(1+2D)}\right.$

$= \frac{-1}{4}\frac{2D-11 \sin 5x}{(4D^2-121)} + \frac{1}{4}\frac{1+2D}{1-4D^2} \sin x$

$$= \frac{(2D-11)\sin 5x}{-4(-100-121)} + \frac{1}{4}\frac{(1+2D)\sin x}{1+4}$$

$$= \frac{20\sin 5x - 11\sin 5x}{884} + \frac{\sin x + 2D\sin x}{20}$$

$$= \frac{120\cos 5x - 11\sin 5x}{884} + \frac{\sin x + 2\cos x}{20}$$

$$y = CF + PI$$

$$y = Ae^{x} + Be^{3x} + \frac{10\cos 5x - 11\sin 5x}{884} + \frac{\sin x + 2\cos x}{20}$$

3. solve $(D^2 - 4D + 4)y = 3x^2 e^{2x}\sin 2x$

soln :-

to find CF

the AE is $m^2 - 4m + 4 = 0$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$m = 2, 2$$

$\therefore$ Real & equal

CF is $(c_1 x + c_2)e^{2x}$

PI $\quad PI = \frac{1}{D^2 - 4D + 4} 3x^2 e^{2x}\sin 2x$

$$= 3 \cdot e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2 \sin 2x$$

$$= 3e^{2x} \frac{1}{D^2 + 4 + 4D - 4D - 8 + 4} x^2 \sin 2x$$

$$= 3e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

$= 3e^{2x} \int \left(\int x^2 \sin 2x \, dx \right) dx$

first,
we find $\int x^2 \sin 2x \, dx$

$u = x^2 \qquad dv = \sin 2x \, dx$

$du = 2x \, dx \qquad v = \frac{-\cos 2x}{2}$

Integrating by parts,

$$\int x^2 \sin 2x \, dx = \frac{-x^2 \cos 2x}{2} + \int \frac{\cos 2x}{2} \cdot 2x \, dx$$

$$= \frac{-x^2 \cos 2x}{2} + \int x \cos 2x \, dx$$

$u_1 = x \qquad dv_1 = \cos 2x \, dx$

$du_1 = dx \qquad v_1 = \frac{\sin 2x}{2}$

$$= \frac{-x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} dx$$

$$= \frac{-x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} - \frac{2}{2} \cos 2x$$

$$= 3e^{2x} \int \frac{x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} - \cos 2x$$

$$= 3e^{2x} - \frac{x^2}{4} \sin 2x - \frac{x \cos 2x}{2} + \frac{3}{8} \sin 2x$$

$$\left[\begin{array}{l} (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots \\ (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots \\ (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots \\ (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots \end{array} \right]$$

15/9/2020

UNIT-II

LINEAR EQNS WITH VARIABLE COEFFICIENTS :-

A homogenous linear eqn is of the form,

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + P_n y = x \longrightarrow (1)$$

where $P_1, P_2, P_3 \dots P_n$ are constants & x a $f_n(x)$.

we have,

$$\boxed{xD = \theta}, \boxed{x^2 D^2 = \theta(\theta-1)}$$

$$\boxed{x^3 D^3 = \theta(\theta-1)(\theta-2)} \text{ \& so on} \longrightarrow (2)$$

Putting $x = e^z$ & we denote θ as the operator $\frac{d}{dz}$

then (1) becomes,

$$f(\theta) y = z \longrightarrow (3)$$

which is linear eqn with constant coefficients.

PRBLMS :-

1. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = x^2$ solve it.

soln :-

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = x^2 \longrightarrow (1)$$

$$(x^2 D^2 - 3xD + 4)y = x^2$$

Put $x = e^z$ (ie) $z = \log x$ & θ denote by $\frac{d}{dz}$

$$(\theta(\theta-1) - 3\theta + 4)y = e^{2z}$$

$$(\theta^2 - \theta - 3\theta + 4)y = e^{2z}$$

$$(\theta^2 - 4\theta + 4)y = e^{2z} \longrightarrow (2)$$

$$y = CF + PI$$

To find CF

$$(\theta^2 - 4\theta + 4)y = 0$$

The AE is $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$\begin{array}{c|c} & 4 \\ \hline -2 & 2 \\ \hline m & m \end{array}$

The roots are real & equal.

$$CF = (c_1 z + c_2)e^{2z}$$

$$CF = (c_1(\log x) + c_2)e^{2z}$$

To find PI

$$PI = \frac{1}{\theta^2 - 4\theta + 4} e^{2z}$$

$$f(\theta) = \theta^2 - 4\theta + 4$$

$$f'(\theta) = 2\theta - 4 \qquad \theta = 2$$

$$f(2) = 4 - 4$$

$$= 0$$

$$f''(\theta) = 2$$

$$f''(2) = 2 \neq 0$$

$$PI = z^2 \cdot \frac{1}{2} e^{2z} \left[\frac{1}{f(\theta)} e^{az} = z^2 \frac{1}{f''(a)} e^{az} \text{ if } f''(\theta) \neq 0 \right]$$

$$PI = \frac{(\log x)^2}{2} \cdot x^2$$

$$PI = \frac{(\log x)^2}{2} x^2$$

$$y = (c_1(\log x) + c_2)x^2 + \frac{x^2}{2}(\log x)^2$$

2. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x \log x$

Soln:-

Gn, $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x \log x \longrightarrow (1)$

$$(x^2 D^2 + xD + 1)y = x \log x$$

Put $x = e^z$ (ie) $z = \log x$ & θ denoted by $\frac{d}{dz}$

we have $xD = \theta$, $x^2D^2 = \theta(\theta-1)$

$$(\theta(\theta-1)+\theta+1)y = ze^{z}$$

$$(\theta^2-\theta+\theta+1)y = ze^{z}$$

$$(\theta^2+1)y = ze^{z} \longrightarrow (2)$$

$y = CF + PI.$

To find CF $\quad (\theta^2+1)y = 0$

the AE is $m^2+1=0$

$$m^2 = \pm 1$$

$$m = \pm i$$

$\therefore$ Imaginary & occur in conjugate pairs

$$CF = c_1 \cos z + c_2 \sin z$$

To find PI

$$PI = \frac{1}{\theta^2+1} ze^{z}$$

$$= e^{z} \frac{1}{(\theta+1)^2+1} z$$

$$= e^{z} \frac{1}{\theta^2+2\theta+2} z.$$

$$= e^{z} \frac{1}{2\left(1+\frac{\theta^2}{2}+\theta\right)} z.$$

$$= e^{z} \frac{1}{2\left(1+\frac{\theta^2+2\theta}{2}\right)} z$$

$$= \frac{e^{z}}{2}\left(1+\frac{\theta^2+2\theta}{2}\right)^{-1} z$$

$$= \frac{e^{z}}{2}\left(1-\frac{\theta^2+2\theta}{2}\right) z$$

$$= \frac{e^{z}}{2}\left[z - \frac{1}{2}(\theta^2+2\theta)z\right)$$

$$= \frac{e^{z}}{2}\left[z - \frac{1}{2}\theta^2(z) + 2\theta(z)\right]$$ doubt

$$= \frac{e^{z}}{2}\left[z-0-1\right]$$

$$P.I = \frac{e^{z}}{2}(z-1)$$

$y = CF + PI$

$= (c_1 \cos z + c_2 \sin z) + \frac{e^z}{2}(z-1)$

$= (c_1 \cos(\log x) + c_2 \sin(\log x)) + \frac{x}{2}(\log x - 1)$

Pr 1. solve $x^2 \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} - 3y = x^2$

soln:

Given that

$$x^2 \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} - 3y = x^2$$

$$(x^2 D^2 + xD - 3)y = x^2 \longrightarrow (1)$$

Put $x = e^z$ (ie) $z = \log x$ & θ denoted by $\frac{d}{dz}$

we have, $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$

$$(\theta(\theta-1) + \theta - 3)y = e^{2z}$$

$$(\theta^2 - \theta + \theta - 3)y = e^{2z}$$

$$(\theta^2 - 3)y = e^{2z}$$

To find CF

$$(\theta^2 - 3)y = 0$$

The AE is, $m^2 - 3 = 0$

$m^2 = 3$

$m = \pm 3$

$\therefore$ Real & Distinct

$$CF = c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}$$

To find PI

$$PI = \frac{1}{\theta^2 - 3} e^{2z}$$

$$= \frac{1}{4-3} e^{2z} \quad (\theta = 2)$$

$$= \frac{1}{1} e^{2z}$$

$$PI = e^{2z}$$

$$y = c_1 e x^{\sqrt{3}} + c_2 e x^{-\sqrt{3}} + e^{2z}$$

$$= c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}} + x^2$$

Solve, $x^3 \cdot \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x$.

Soln :-

$$(x^3D^3 + 3x^2D^2 + xD + 1)y = x + \log x$$

Putting $z = \log x \quad D = \frac{d}{dz}$

$$(\theta(\theta-1)(\theta-2) + 3(\theta-1) + \theta + 1)y = e^z + z$$

$$(\theta(\theta^2 - 2\theta - \theta + 2) + 3(\theta^2 - \theta) + \theta + 1)y = e^z + z$$

$$\theta(\theta^2 - 3\theta + 2) + 3\theta^2 - 3\theta + \theta + 1)y = e^z + z$$

$$(\theta^3 - 3\theta^2 + 2\theta + 3\theta^2 - 3\theta + \theta + 1)y = e^z + z$$

$$(\theta^3 + 1)y = e^z + z$$

To find CF

The AE is $m^3 + 1 = 0$

$m^3 = -1$

$m = -1 \quad (\text{or}) \quad m = \frac{1 \pm \sqrt{3}i}{2}$

$$CF = c_1 e^{-z} + e^{\frac{z}{2}}\left(c_2 \cos \frac{\sqrt{3}}{2} z + c_3 \sin \frac{\sqrt{3}}{2} z\right)$$

$$PI = \frac{1}{D^3+1}(e^z + z)$$

$$= \frac{1}{2}e^z + (1 - D^3)z$$ Doubt

$$= \frac{x}{2} + \log x$$

$y = CF + PI$

$$= c_1 x^{-1} + \sqrt{x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + \frac{x}{2} + \log x.$$

$x^2 \frac{d^2y}{dx^2} + 8x \cdot \frac{dy}{dx} + 12y = x^4$.

soln :

$$x^2 \frac{d^2y}{dx^2} + 8x \cdot \frac{dy}{dx} + 12y = x^4$$

Putting $x = e^z$, $z = \log x$ & $D = \frac{d}{dz}$.

$(x^2D^2 + 8xD + 12)y = e^{4z}$

$(\theta(\theta-1) + 8\theta + 12)y = e^{4z}$

$(\theta^2 - \theta + 8\theta + 12)y = e^{4z}$

$(\theta^2 + 7\theta + 12)y = e^{4z}$

<u>To find CF</u>

The AE is $m^2 + 7m + 12 = 0$

$m^2 + 4m + 3m + 12 = 0$

$m(m+4) + 3(m+4) = 0$

$(m+3)(m+4) = 0$

$m = -3, -4$

CF is $c_1x^{-3} + c_2x^{-4}$ doubt e^{-}?

<u>To find PI</u>

$$PI = \frac{1}{\theta^2 + 7\theta + 12} e^{4z}$$

$$= \frac{1}{16 + 7\theta + 12} e^{4z} \quad (\theta = 4)$$

$$= \frac{1}{7\theta + 28} e^{4z} \quad \begin{matrix}\theta = 4 \\ (7\times4 = 28)\end{matrix}$$

$$= \frac{1}{28 + 28} \cdot e^{4z}$$

$$= \frac{1}{56} e^{4z}$$

$$PI = \frac{e^{4z}}{56}$$

$$y = CF + PI$$

$$= c_1e^{-3z} + c_2 e^{-4z} + \frac{e^{4z}}{56}$$

$$= c_1x^{-3} + c_2x^{-4} + \frac{x^4}{56}$$

4) $x^2 \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + 2y = x^2$

<u>soln:-</u>

$(x^2D^2 + xD + 2)y = e^{2z}$

$(\theta(\theta-1) + \theta + 2)y = e^{2z}$

$(\theta^2 - \cancel{\theta} + \cancel{\theta} + 2)y = e^{2z}$

$$(\theta^2+2)y = e^{2z}$$

To find CF

The AE is $m^2+2=0$

$m^2 = -2$

$m = -\sqrt{2}i$

$CF = A\cos\sqrt{2}z + B\sin\sqrt{2}z$

To find PI

$$PI = \frac{1}{\theta^2+2} e^{2z}$$

$$= \frac{1}{4+2} e^{2z}$$

$$= \frac{e^{2z}}{8}$$

$$PI = \frac{e^{2z}}{8}$$

$y = CF + PI$

$$y = A\cos\sqrt{2}z + B\sin\sqrt{2}z + \frac{e^{2z}}{8}$$

$$y = A\cos(\sqrt{2}\log x) + B(\sin\sqrt{2}\log x) + \frac{x^2}{8}$$

EG :- 2

solve $\frac{x^2 d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$

soln :-

G.T. $\frac{x^2 d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$

$$(x^2D^2 + 3D + 1)y = \frac{1}{(1-x)^2} \longrightarrow (1)$$

Put $x = e^z$ & $z = \log x$ & θ denoted by $\frac{d}{dx}$

we have $xD = \theta$, $x^2D^2 = \theta(\theta-1)$

(1) $\Rightarrow$

$$(\theta(\theta-1) + 3\theta + 1)y = \frac{1}{(1-e^z)^2}$$

$$(\theta^2 - \theta + 3\theta + 1)y = \frac{1}{(1-e^z)^2}$$

$$(\theta^2 + 2\theta + 1)y = \frac{1}{(1-e^z)^2} \longrightarrow (2)$$

which is linear eqn with constant coefficient.

the soln of ② is $y = CF + PI$

<u>To find CF</u>

the AE is $m^2 + 2m + 1 = 0$

$$(m+1)^2 = 0$$

$$m = -1, -1$$

∴ Real & Two of them are equal

$$CF = e^{-z}(c_1 z + c_2)$$

$$CF = \frac{1}{x}(c_1 \log x + c_2)$$

<u>To find PI</u>

$PI = \frac{1}{\theta - \alpha}\, \alpha$ where x is function of x

$$\left[\frac{1}{\theta - \alpha} X = x^{\alpha} \int x^{-\alpha - 1} X\, dx\right] \text{ formula}$$

$$PI = \frac{1}{\theta^2 + 2\theta + 1} \cdot \frac{1}{(1-x)^2}$$

$$= \frac{1}{(\theta+1)^2} \cdot \frac{1}{(1-x)^2}$$

$$= \frac{1}{\theta+1} \cdot \frac{1}{\theta+1} \cdot \frac{1}{(1-x)^2}$$

$$= \frac{1}{\theta+1}\left\{\frac{1}{\theta - (-1)} \cdot \frac{1}{(1-x)^2}\right\}$$ $\alpha = -1$, $X = \frac{1}{(1-x)^2}$

$$= \frac{1}{\theta+1}\left\{x^{-1} \int x^{1-1} \frac{1}{(1-x)^2}\, dx\right\}$$

$$= \frac{1}{\theta+1}\left\{\frac{1}{x} \int \frac{1}{(1-x)^2}\, dx\right\}$$

$$= \frac{1}{\theta+1}\left\{\frac{1}{x} - \frac{(1-x)^{-2+1}}{-2+1}\right\}$$

$$= \frac{1}{\theta+1}\left\{\frac{1}{x} - \frac{(1-x)^{-1}}{-1}\right\}$$

$$x^{2} \int x^{-\alpha - 1} x\, dx = \frac{1}{\theta+1}\left\{\frac{1}{x} \cdot \frac{1}{(1-x)}\right\}$$

$$= \frac{1}{0-(-1)} \frac{1}{x(1-x)} \text{ (formula)}$$

$$= x^{-1} \int x^{1-1} \frac{1}{x(1-x)} dx$$

$$= \frac{1}{x} \int \frac{1}{x(1-x)} dx$$

$$= \frac{1}{x} \int d\left(\log \frac{x}{1-x}\right)$$

$$\left[d\left(\log \frac{x}{1-x}\right) = \frac{1}{\frac{x}{1-x}} \left(\frac{1-x-(x}{(1-x)^2} \right.\right.$$

$$= \frac{1-x}{x} \quad 1-x$$

$$PI = \frac{1}{x} \log \frac{x}{1-x}$$

$$y = CF + PI$$

$$y = e^{-z}(c_1 + c_2 z) + \frac{1}{x} \log \frac{x}{1-x}$$

$$y = \frac{1}{x}(c_1 + c_2 \log x) + \frac{1}{x} \log \frac{x}{1-x}.$$

Exa :-

3. solve $\frac{x^2 d^2 y}{dx^2} + 4x \cdot \frac{dy}{dx} + 2y = e^x.$

soln :-

G.T $\frac{x^2 d^2 y}{dx^2} + 4x \cdot \frac{dy}{dx} + 2y = e^x.$

$$(x^2D^2 + 4xD + 2)y = e^x \longrightarrow (1)$$

Put $x = e^z$ in $z = \log x$ & θ denoted by $\frac{d}{dz}$

we have $xD = \theta$; $x^2D^2 = \theta(\theta - 1)$

$$(1) \Rightarrow (\theta(\theta-1) + 4\theta + 2)y = e^x$$

$$(\theta^2 - \theta + 4\theta + 2)y = e^x$$

$$(\theta^2 + 3\theta + 2)y = e^x$$

To find CF

The AE is $m^2 + 3m + 2 = 0$

$$m^2 + m + 2m + 2 = 0$$

$$m(m+1) + 2(m+1) = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

$\therefore$ Real & distinct

The CF is $c_1 e^{-z} + c_2 e^{-2z}$

To find PI

$$PI = \frac{1}{\theta^2+3\theta+2} e^x \Rightarrow \frac{1}{(\theta+1)(\theta+2)} e^x$$

$$\frac{A}{\theta+1} + \frac{B}{\theta+2}$$

$$1 = A(\theta+2) + B(\theta+1)$$

$$A = 1 \quad B = -1$$

$$\frac{A(\theta+2) + B(\theta+1)}{(\theta+1)(\theta+2)}$$

$$\frac{1}{(\theta+1)(\theta+2)} = \left[\frac{1}{(\theta+1)} - \frac{1}{\theta+2}\right] e^x$$

formula

$$= \frac{1}{\theta+1} e^x \left\{\frac{1}{\theta-(-2)} e^x\right\}$$

(formula) $\left(\frac{1}{\theta-\alpha} x = x^{\alpha} \int x^{-\alpha-1} x \, dx\right)$

$$= \frac{1}{\theta+1} e^x - \left\{x^{-2} \int x^{2-1} e^x dx\right.$$

$$= \frac{1}{\theta+1} e^x - \left\{\frac{1}{x^2} \cdot \int \underset{uv}{x e^x} dx\right\}$$

$$= \frac{1}{\theta+1} e^x - \left\{\frac{1}{x^2} x \cdot e^x - \int e^x dx\right\}$$

$$= \frac{1}{\theta+1} e^x - \left\{\frac{1}{x^2}(xe^x - e^x)\right\}$$

$$= \frac{1}{\theta-(-1)} e^x - \left\{x^{-2}(xe^x - e^x)\right\}$$

formula

$$= x^{-1} \int x^{1-1} e^x dx - x^{-2}(xe^x - e^x)$$

$$= \frac{1}{x} \int e^x dx - x^{-2}(xe^x - e^x)$$

$$= \frac{1}{x} e^x - x^{-2}(xe^x - e^x)$$

$$= \cancel{x} \, x^{-1} e^x - x^{-2}(xe^x - e^x)$$

$$= \cancel{x^{-1} e^x} - \cancel{x^{-1} e^x} + x^{-2} e^x$$

$$PI = x^{-2} e^x$$

$$y = CF + PI$$

$$= c_1 e^{-z} + c_2 e^{-2z} + x^{-2} e^x$$

$$= c_1 x^{-1} + c_2 x^{-2} + x^{-2} e^x.$$

15. solve $(x^3D^3+3x^2D^2+xD+1)y=\sin(\log x)$

G.T $(x^3D^3+3x^2D^2+xD+1)y=\sin(\log x) \longrightarrow (1)$

Put $x=e^z$ & $z=\log x$ & θ denoted by $\frac{d}{dz}$

we have $x^3D^3=\theta(\theta-1)(\theta-2)$

$x^2D^2=\theta(\theta-1)$

$xD=\theta$

$(1) \Rightarrow (\theta(\theta-1)(\theta-2)+3(\theta(\theta-1)+\theta+1)y=\sin z$

$(\theta(\theta^2-2\theta-\theta+2)+3(\theta^2-\theta)+\theta+1)y=\sin z$

$(\theta^3-3\theta^2+2\theta+3\theta^2-3\theta+\theta+1)y=\sin z$

$(\theta^3+1)y=\sin z \longrightarrow (2)$

soln of (2) is $y=CF+PI$

<u>To find CF</u>

the AE is $m^3-2m+3=0$

$m^3+1=0$

$m^3=-1$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & 0 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

$(m+1)(m^2-m+1)=0$

$m=-1, \quad \frac{+1\pm\sqrt{1-4}}{2}$

$m=-1, \quad \frac{+1\pm\sqrt{-3}}{2}$

$m=-1, \quad \frac{1\pm\sqrt{3}i}{2}$

$$CF=c_1x^{-1}+e^{\frac{1}{2}z}\left[c_2\cos\frac{\sqrt{3}}{2}z+c_3\sin\frac{\sqrt{3}}{2}z\right]$$

<u>To find PI</u>

$PI=\frac{1}{\theta^3+1}\cdot\sin z(\log x)$

$PI=\frac{1}{\theta^3-(-1)}\sin\log x$

$=x^{-1}\int x^{1-1}\sin(\log x)\,dx$

$=\frac{1}{x}\int\sin(\log x)\,dx$

$$= \frac{1}{(\theta+1)(\theta^2-\theta+1)} \sin z$$

$$= \frac{1}{(\theta+1)(-\theta)} \sin z$$

$$= \frac{1}{-(\theta^2+\theta)} \sin z$$

$$= \frac{1}{-(-1+\theta)} \sin z \qquad (1+\theta)$$

$$= \frac{1}{1-\theta} \sin z$$

$$= \frac{1+\theta}{(1-\theta)(1+\theta)} \sin z$$

$$= \frac{1+\theta}{1-\theta^2} \sin z$$

$$= \frac{1+\theta}{2} \sin z$$

$$= \frac{1}{2}\left[\sin z + \theta \sin z\right]$$

$$= \frac{1}{2}\left[\sin \log x + \frac{d}{dz} \sin z\right] \qquad z = \log x.$$

$$= \frac{1}{2}\left[\sin \log x + \cos \log x\right]$$

$$y = CF + PI$$

$$= c_1 \frac{1}{x} + \sqrt{x}\left(c_2 \cos \frac{\sqrt{3}}{2}(\log x) + c_3 \sin \frac{\sqrt{3}}{2}(\log x)\right) + \frac{1}{2}\left(\sin \log x + \cos \log x\right)$$

EQUATIONS REDUCIBLE TO THE LINEAR EQNS :

Consider an eqn of the form,

$$(a+bx)^n \frac{d^n y}{dx^n} + (a+bx)^{n-1} P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = X$$

where $P_1, P_2 \dots P_n$ are constants & X is any fun of x.

PRBLM

1. solve $(5+2x)^2\frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 6x$

soln

$$(5+2x)^2\frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 6x \longrightarrow (1)$$

$$u = 5+2x \Rightarrow x = \frac{u-5}{2}$$

$$\frac{du}{dx} = 2$$

x, ÷ by du

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} = 2\cdot\frac{dy}{du}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(2\cdot\frac{dy}{du}\right)$$

× by ÷ du

$$= 2\cdot\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = 2^2\cdot\frac{d^2y}{du^2}$$

$$(1) \Rightarrow u^2 4\frac{d^2y}{du^2} - 12u\frac{dy}{du} + 8y = 3(u-5)$$

∴ which is a linear coefficient eqn with variable coefficient.

$$(4u^2D^2 - 12uD + 8)y = 3u - 15 \longrightarrow (2)$$

Put $u = e^z$; $\log u = z$ θ denoted by $\frac{d}{dz}$

$$uD = \theta,\quad u^2D^2 = \theta(\theta-1)$$

$$(2) \Rightarrow (4\theta(\theta-1) - 12\theta + 8)y = 3e^z - 15$$

$$(4\theta^2 - 4\theta - 12\theta + 8)y = 3e^z - 15$$

$$(4\theta^2 - 16\theta + 8)y = 3e^z - 15$$

$$4(\theta^2 - 4\theta + 2)y = 3(e^z - 5)$$

$$(\theta^2 - 4\theta + 2)y = \frac{3}{4}(e^z - 5) \longrightarrow (3)$$

This linear eqn is Constant Coefficient

$$y = CF + PI$$

To find CF

The AE is $m^2 - 4m + 2 = 0$

$$m^2 - 2m - 2m + 2 = 0$$

$$m(m-2) - 2(m-1) = 0$$

$$(m-2)(m-2) = 0$$

$$m = 2, 2$$

real & equal

$$= \frac{4 \pm \sqrt{16-8}}{2}$$

Doubt

$$= \frac{4 \pm \sqrt{8}}{2}$$

$$= \frac{4 \pm 2\sqrt{2}}{2}$$

$$= \frac{2(2 \pm \sqrt{2})}{2}$$

$$m = 2 \pm \sqrt{2}$$

$$CF = e^{2u}\left(c_1 e^{\sqrt{2}u} + B_2 e^{-\sqrt{2}u}\right)$$

To find PI

$$PI = \frac{3}{4(\theta^2 - 4\theta + 2)}(e^u - 5)$$

$$= \frac{3e^u}{4(\theta^2 - 4\theta + 2)} - \frac{15}{4(\theta^2 - 4\theta + 2)}$$

Sub θ value

$$= \frac{3e^u}{-4} - \frac{15}{8\left(1 + \frac{\theta^2 - 4\theta}{2}\right)}$$

outside

$$= \frac{-3}{4}e^u - \frac{15}{8}\left(1 + \frac{\theta^2 - 4\theta}{2}\right)^{-1}$$

$$= \frac{-3}{4}e^u - \frac{15}{8}(1) \text{ (omitting all the terms)}$$

$$PI = \frac{-3}{4}e^u - \frac{15}{8}$$

$$y = CF + PI.$$

$$= x^2\left(c_1 x^{\sqrt{2}} + c_2 x^{-\sqrt{2}}\right) - \frac{3}{4}e^u - \frac{15}{8}$$

$$= u^2\left(c_1 u^{\sqrt{2}} + c_2 u^{\sqrt{2}}\right) \cdot \frac{-3}{4}u - \frac{15}{8}$$

Ex: 4 W-Y-4

$$= (5x+2x)^2 \left(c_1 (5x+2x)^{1/2} + c_2 (5+2x)^{-1/2} - \frac{3}{4}(5+x) \frac{-15}{8} \right.$$

Eg. 2

solve $(1+x^2)^3 \frac{d^2y}{dx^2} + 2x(1+x^2)^2 \frac{dy}{dx} + (1+x^2)y = 3x$

soln :

G.T. $(1+x^2)^3 \frac{d^2y}{dx^2} + 2x(1+x^2)^2 \frac{dy}{dx} + (1+x^2)y = 3x$

$$x = \tan\theta$$

$$\frac{dx}{d\theta} = \sec^2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{dy}{d\theta} \frac{1}{\sec^2\theta}$$

$$\frac{dy}{dx} = \cos^2\theta \frac{dy}{d\theta}$$

$$\frac{dy}{d\theta} = \frac{1}{\cos^2\theta} \frac{dy}{dx}$$

$$= \sec^2\theta \frac{dy}{dx}$$

$$= (1+\tan^2\theta) \frac{dy}{dx}$$

$$\frac{dy}{d\theta} = (1+x^2) \frac{dy}{dx}$$

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}\left(\frac{dy}{d\theta}\right)$$

$$= \frac{d}{d\theta}\left((1+x^2) \frac{dy}{dx} \right)$$

$$= \frac{d}{dx}\left((1+x^2) \frac{dy}{dx} \right) \cdot \frac{dx}{d\theta}$$

$$= \left((1+x^2) \frac{d^2y}{dx^2} + 2x \right) (1+x^2)^2 \frac{dy}{dx}$$

$$(1+x^2) \frac{d^2y}{d\theta^2} = (1+x^2)^3 \cdot \frac{d^2y}{dx^2} + 2x(1+x^2)^2 \cdot \frac{dy}{dx}$$

H.W

1. Find CF of $(D^4 - 4D^3 + 8D^2 - 8D + 4)y = 0$

<u>soln:</u>

$$m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$$

$$(m^2 - 2m + 2)^2 = 0$$

$$= \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$

$m = (1 \pm i)$ is twice

2. solve $(D^4 + D^3 + D^2)y = 5x^2 + \cos x$

<u>soln:-</u>

The AE is $m^4 + m^3 + m^2 = 0$

$$m = 0, \quad m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$CF = A + Bx + e^{-x/2}\left(C\cos\frac{\sqrt{3}}{2}x + D\sin\frac{\sqrt{3}}{2}x\right)$$

$$PI = \frac{5x^2}{D^2(D^2+D+1)}$$

$$= \frac{1}{D^2}(1+D+D^2)^{-1} 5x^2$$

$$= \frac{1}{D^2}\left\{\left(1-(D+D^2) + (D+D^2)^2 - (D+D^2)^3 + (D+D^2)^4\right\}\right.$$

$f(x^2$

$$= \frac{1}{D^2}(1 - D + D^3 - D^4)\, 5x^2$$

$$= \frac{1}{D^2}5x^2 - \frac{1}{D}5x^2 + (D - D^2)5x^2$$

$$= \int\int 5x^2 (dx)^2 - 5\int x^2 dx + (10x - 10)$$

$$= \frac{5x^4}{12} - \frac{5x^3}{3} + 10x - 10$$

$$PI = \frac{\cos x}{D^2(D^2+D+1)}$$

$$= \frac{1}{-1(-1+D+1)} \cos x$$

$$= \frac{-1}{D} \cos x$$

$$= -\sin x$$

$$y = A + Bx + e^{-2}\left(C\cos\frac{\sqrt{3}}{2}x + D\sin\frac{\sqrt{3}}{2}x\right) +$$

$$\frac{5x^4}{12} - \frac{5x^2}{3} + 10x - 10 - \sin x.$$

3. solve $x - yp = ap^2$

<u>soln</u> :-

Gn $(x - yp) = ap^2$

$$x = yp + ap^2 \longrightarrow ①$$

diff w. r. to y

$$\frac{dx}{dy} = p + y \cdot \frac{dp}{dy} + 2ap\frac{dp}{dy}$$

$$p - \frac{1}{p} + (y + 2ap)\frac{dp}{dy} = 0$$

$$\frac{p^2-1}{p} + (y + 2ap)\frac{dp}{dy} = 0$$

$$(y + 2ap)\frac{dp}{dy} = -\left(\frac{p^2-1}{p}\right)$$

$$\left(\frac{1-p^2}{p}\right)\frac{dy}{dp} = y + 2ap$$

$$\frac{dy}{dp} = \frac{p(y+2ap)}{1-p^2}$$

$$\frac{dy}{dp} = \frac{p}{1-p^2}y + \frac{2ap^2}{1-p^2}$$

$$\frac{dy}{dp} = \frac{-p}{p^2-1}y + \frac{2ap^2}{1-p^2}$$

$$\frac{dy}{dp} + \frac{p}{p^2-1}y = \frac{2ap^2}{1-p^2}$$

This linear eqn,

$$P = \frac{p}{p^2-1} \quad Q = \frac{2ap^2}{1-p^2}$$

<u>to find I.F</u> $e^{\int P dp}$

$$IF = e^{\int P dp}$$

$$= e^{\int \frac{p}{p^2-1} dp}$$

$$= e^{\int \frac{2p}{2(p^2-1)} dp}$$

$$= e^{\frac{1}{2} \log (p^2-1)}$$

$$= e^{\log (p^2-1)^{1/2}}$$

$$IF = (p^2-1)^{\frac{1}{2}}$$

The soln is $y\, e^{\int P dp} = \int Q \cdot e^{\int P dp} dp + c$

$$y(p^2-1)^{\frac{1}{2}} = \int \frac{ap^2}{1-p^2} (p^2-1)^{\frac{1}{2}} dp$$

$$y \cdot (p^2-1)^{\frac{1}{2}} = \int \frac{ap^2}{1-p^2} (p^2-1)^{\frac{1}{2}} dp$$

$$\sqrt{1-p^2}\, x = p(c + a \sin^{-1} p) \;\&\; x - yp = ap^2$$

solve $(px-y)(py+x) = 2p$

<u>soln</u>

Put $X = x^2, \; Y = y^2$

$2x\,dx = dX \qquad 2y\,dy = dY$

$$\frac{dY}{dX} = \frac{2y\,dy}{2x\,dx}$$

$$P = \frac{y}{x} p$$

$$p = \frac{x}{y} P$$

$$\left(\frac{x}{y} Px - y\right)\left(\frac{x}{y} Py + x\right) = \frac{2x}{y} P$$

$$\left(\frac{x^2P-y^2}{y}\right)(xp+x) = \frac{2x}{y}P$$

$$\left(\frac{x^2P-y^2}{y}\right)(xp+x) = \frac{2x}{y}P$$

$$\cancel{x/y}\ (x^2P-y^2)(P+1) = 2\frac{x}{y}P$$

$$(xP-Y)(P+1) = 2P$$

$$xp - Y = \frac{2P}{P+1}$$

$$-Y = \frac{2P}{P+1} - xP$$

$$Y = xP - \frac{2P}{P+1}$$

This is of Claurits eqn,

$y = Px + f(P)$ we replace P by c

$Y = xc - \frac{2a}{c+1}$ is the general soln.

6. Solve $(x^2 - 2xy + 3y^2)dx + (4y^3 + 6xy - x^2)dy = 0$

soln :-

The eqn is of the form $mdx + Ndy = 0$

$m = x^2 - 2xy + 3y^2$ $\qquad$ $N = 4y^3 + 6xy - x^2$

$\frac{\partial M}{\partial y} = -2x + 6y$ $\qquad$ $\frac{\partial N}{\partial x} = 6y - 2x$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence ① is an exact diff eqn

The soln of ① is $\int M dx + \int N dy = 0$

$\int m dx = \int x^2 - 2xy + 3y^2\, dx$

$= \frac{x^3}{3} - 2\frac{x^2}{2}\cdot y + 3xy^2$

$= \frac{x^3}{3} - x^2 y + 3xy^2$

$\int N dy = \int 4y^3 dy$

$= 4 \cdot \frac{y^4}{4}$.

$$\int M\,dx + \int N\,dy = \frac{x^3}{3} - x^2y + 3xy^2 + y^4$$

EQN REDUCIBLE TO LINEAR EQN

Eg:3 solve,

$$x^2 \cdot \frac{d^2y}{dx^2} + (4x^2+6x)\frac{dy}{dx} + (3x^2+12x+6)\,y = 0$$

soln:

eqn that, $x^2\frac{d^2y}{dx^2} + (4x^2+6x)\frac{dy}{dx} + (3x^2+12x+6)\,y = 0 \longrightarrow (1)$

we have $\boxed{z = yx^3}$, w.r. ~~diff~~

$$\frac{dz}{dx} = y\cdot 3x^2 + x^3\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{x^3}\left(\frac{dz}{dx} - 3x^2y\right)$$

$$\frac{d^2z}{dx^2} = 3\left[y\cdot 2x + x^2\frac{dy}{dx}\right] + 3x^2\frac{dy}{dx} + x^3\cdot\frac{d^2y}{dx^2}$$

$$\frac{d^2z}{dx^2} = 6xy + 3x^2\frac{1}{x^3}\left(\frac{dz}{dx} - 3x^2y\right) + 3x^2\cdot\frac{1}{x^3}\left(\frac{dz}{dx} - 3\right. + x^3\frac{d^2y}{dx^2}$$

$$= 6xy + \frac{3}{x}\left(\frac{dz}{dx} - 3x^2y\right) + \frac{3}{x}\left(\frac{dz}{dx} - 3x^2y\right) + x^3\cdot\frac{d^2y}{dx^2}$$

$$\frac{d^2z}{dx^2} = 6xy + \frac{6}{x}\left(\frac{dz}{dx} - 3x^2y\right) + x^3\cdot\frac{d^2y}{dx^2}$$

$$x^3\frac{d^2y}{dx^2} = \frac{d^2z}{dx^2} - 6xy - \frac{6}{x}\left(\frac{dz}{dx} - 3x^2y\right)$$

$$x^2\cdot\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2z}{dx^2} - 6y - \frac{6}{x^2}\left(\frac{dz}{dx} - 3x^2y\right)$$

(1) $\Rightarrow$

$$\frac{1}{x}\cdot\frac{d^2z}{dx^2} - 6y - \frac{6}{x^2}\left(\frac{dz}{dx} - 3x^2y\right) + (4x^2+6x)\frac{1}{x^3}\left(\frac{dz}{dx} - 3x^2y\right)$$

$$+ (3x^2+12x+6)\,y = 0$$

$\frac{1}{x}\frac{d^2z}{dx^2} - 6y - \frac{6}{x^2}\frac{dz}{dx} + 18x^2y^2 + \frac{1}{x^2}(4x+6)\left(\frac{dz}{dx} - 3x^2y\right) +$

$3x^2y + 12xy + 6y = 0$

$= \frac{1}{x}\left(\frac{d^2z}{dx^2} - \frac{6}{x^2}\frac{dz}{dx} + 18y + \frac{4}{x}\frac{dz}{dx} - 12xy + \frac{6}{x^3}\cdot\frac{dz}{dx} - 18y + 3x^2y\right.$

$+ 12xy = 0$

$= \frac{1}{x}\frac{d^2z}{dx^2} + \frac{4}{x}\frac{dz}{dx} + 3x^2y = 0$

$= \frac{1}{x}\left\{\frac{d^2z}{dx^2} + 4\frac{dz}{dx} + 3x^3y\right\} = 0$

$$\frac{d^2z}{dx^2} + 4\frac{dz}{dx} + 3x^3y = 0$$

$$(D^2+4D+3)z = 0 \quad (z = x^3y)$$

To find CF

The AE is $m^2+4m+3=0$ (RHS 0 so no need to find PI)

$m^2+m+3m+3=0$

$m(m+1)+3(m+1)=0$

$m+1=0 \quad m+3=0$

$m=-1,-3$

$z = c_1e^{-x} + c_2e^{-3x}$

$x^3y = c_1e^{-x} + c_2e^{-3x}$

Eg 2. solve $\cos x\cdot\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} + 4(\cos^3 x)y = 8\cos^5 x$

soln :-

$\cos x\cdot\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} + 4(\cos^3 x)y = 8\cos^5 x \longrightarrow (1)$

Let $z = \sin x$

$\frac{dz}{dx} = \cos x$

$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$

$= \cos x\cdot\frac{dy}{dz}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cos x \cdot \frac{dy}{dz}\right)$$

$$= -\sin x \frac{dy}{dz} + \cos x \cdot \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\sin x \cdot \frac{dy}{dz} + \cos x \cdot \frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -\sin x \frac{dy}{dz} + \cos^2 x \frac{d^2y}{dz^2}.$$

Eg: 2 solve $\cos x \cdot \frac{d^2y}{dx^2} + \sin x \cdot \frac{dy}{dx} + 4(\cos^3 x)y = 8\cos^5 x$.

soln:-

gn that, $\cos x \cdot \frac{d^2y}{dx^2} + \sin x \cdot \frac{dy}{dx} + 4(\cos^3 x)y = 8\cos^5 x \longrightarrow (1)$

let, $z = \sin x$

$$\frac{dz}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$\frac{dy}{dx} = \cos x \cdot \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cos x \cdot \frac{dy}{dz}\right)$$

$$= -\sin x \cdot \frac{dy}{dz} + \cos x \cdot \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\sin x \cdot \frac{dy}{dz} + \cos x \cdot \frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} \quad (\times \div dz)$$

$$= -\sin x \cdot \frac{dy}{dz} + \cos^2 x \frac{d^2y}{dz^2}$$

$(1) \Rightarrow \cos x\left(\cos^2 x \frac{d^2y}{dz^2} \sin x \frac{dy}{dz}\right) + \sin x\left(\cos x \cdot \frac{dy}{dz}\right) + 4(\cos^3 x)y = 8\cos^5 x$

$$\cos^3 x \cdot \frac{d^2y}{dz^2} - \cancel{\frac{\cos x \sin x}{\sin x}} \frac{dy}{dz} + \sin x \cancel{\cos x} \frac{dy}{dz} + 4(\cos^3 x)y = 8\cos^5 x$$

$$\cos^3 x \frac{d^2y}{dz^2} + 4(\cos^3 x)y = 8\cos^5 x$$

common

$$\cos^3 x\left(\frac{d^2y}{dz^2} + 4y\right) = 8\cos^5 x$$

$$\frac{d^2y}{dz^2} + 4y = \frac{8\cos^5 x}{\cos^3 x}$$

$$\frac{d^2y}{dz^2} + 4y = 8\cos^2 x \qquad \sin^2 + \cos^2 = 1$$

$$\frac{d^2y}{dz^2} + 4y = 8(1 - z^2) \qquad 1 - z^2$$

$$(D^2+4)y = 8(1-z^2) \quad \text{value so find PI}$$

by Solving,

The AE is $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

$CF = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$

$CF = A \cos 2x + B \sin 2x$

To find PI

$$PI = \frac{8(1-x^2)}{D^2+4}$$

$$= \frac{8(1-x^2)}{4\left(\frac{D^2}{4}+1\right)} \quad \text{(4 common)}$$

$$= \frac{2(1-x^2)}{1+\frac{D^2}{4}}$$

$$= 2\left(1+\frac{D^2}{4}\right)^{-1}(1-x^2) \quad \text{formula}$$

$$= 2\left(1-\frac{D^2}{4}\right)(1-x^2)$$

$$= \left(2-\frac{D^2}{2}\right)(1-x^2)$$

$$= 2-2x^2-\frac{D^2}{2}+\frac{x^2}{2}(D^2)$$

$$= 2-2x^2+1 \quad \text{diff}$$

$$= 3-2x^2$$

$y = CF + PI$

$$y = A\cos 2x + B \sin 2x + 3-2x^2$$

① solve $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x$

Exer

soln :-

Gn $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x \longrightarrow ①$

$u = x+a \Rightarrow \boxed{x = u-a}$

$\frac{du}{dx} = 1$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{d}{dx}\left(\frac{dy}{du}\right) = \frac{d}{du}\left(\frac{dy}{du}\right)$

$\frac{dy}{dx} = \frac{dy}{dx}$

$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}$

$① \Rightarrow u^2 \cdot \frac{d^2y}{dx^2} - 4u\frac{dy}{du} + 6y = u-a$

$(u^2D^2 - 4uD + 6)y = u-a$

Put $u = e^z \rightarrow z = \log x$ & θ denoted by $\frac{d}{dz}$

$u^2D^2 = \theta(\theta-1) \quad uD = \theta$

$(\theta(\theta-1) - 4\theta + 6)y = u - a$

$(\theta^2 - \theta - 4\theta + 6)y = u - a$

$(\theta^2 - 5\theta + 6)y = u - a$

$y = CF + PI$

<u>To find CF</u>

AE is $m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0$

$m = 3, 2$

$CF = c_1u^2 + c_2u^3$

<u>To find PI</u>

$$PI = \frac{1}{\theta^2 - 5\theta + 6}(e^z - a) \qquad (\theta = 1)$$

$$= \frac{1}{\theta^2 - 5\theta + 6}e^z - \frac{1}{\theta^2 - 5\theta + 6}a \Rightarrow \frac{e^z}{1 - 5\theta + 6} - \frac{1a}{6\left(1 + \frac{\theta^2 - 5\theta}{6}\right)}$$

$$= \frac{e^z}{1 - 5 + 6} - \frac{a}{6}\left[1 + \theta^2 - \frac{5\theta}{6}\right]^{-1} \quad (6 \text{ common})$$

$$= \frac{1}{2}e^z - \frac{a}{6}(1)$$

$$= \frac{e^z}{2} - \frac{a}{6}$$

$$= \frac{u}{2} - \frac{a}{6}$$

$$PI = \frac{3u - a}{6}$$

$$y = c_1(x+a)^2 + c_2(x+a)^3 + \frac{3(x+a) - a}{6}$$

(2) solve $(x+1)^2\frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2 + x + 1$

<u>soln</u>

$$(x+1)^2\frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2 + x + 1 \longrightarrow (1)$$

$u = x + 1 \qquad u - 1 = x$

$\frac{du}{dx} = 1 \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{du}$$

$$= \frac{d}{dx}\left(\frac{dy}{du}\right)$$

$$= \frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx} \quad x, \div du$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}$$

① $\Rightarrow$

$$u^2 \frac{d^2y}{dx^2} - 3u\cdot\frac{dy}{dx} + 4y = u^2 - u + 1$$

$$(u^2D^2 - 3uD + 4)y = u^2 - u + 1$$

$$u = e^z, \quad z = \log x \qquad \theta = \frac{d}{dz}$$

$$u^2D^2 = \theta(\theta - 1), \quad uD = \theta$$

$$(\theta(\theta-1) - 3\theta + 4)y = u^2 - u + 1$$

$$(\theta^2 - \theta - 3\theta + 4)y = u^2 - u + 1$$

$$(\theta^2 - 4\theta + 4)y = u^2 - u + 1$$

$$(\theta^2 - 4\theta + 4)y = e^{2z} - e^z + 1$$

$$y = CF + PI$$

AE is $m^2 - 4m + 4 = 0$

$$(m-2)(m-2) = 0$$

$$m = 2, 2$$

$$CF = (c_1 z + c_2)e^{2z}$$

<u>To find PI</u>

$$PI = \frac{1}{\theta^2 - 4\theta + 4}\cdot e^{2z} - e^z + 1$$

$$= \frac{e^{2z}}{\theta^2 - 4\theta + 4} - \frac{e^z}{\theta^2 - 4\theta + 4} + \frac{1}{\theta^2 - 4\theta + 4}$$

$$= \frac{z^2}{2}e^{2z} - e^z + \frac{1}{(\theta-2)^2}$$

$$= \frac{z^2}{2}e^{2z} - e^z + \frac{1}{-2\left(1-\frac{\theta}{2}\right)2} \quad (-2 \text{ common})$$

$$= \frac{z^2}{2}e^{2z} - e^z - \frac{1}{2}\left[1 - \frac{\theta}{2}\right]^{-2}$$

(omitted) decrease so

$$= \frac{z^2}{2}e^{2z} - e^z - \frac{1}{2}$$

$$= \frac{z^2}{2}e^{2z} - e^z - \frac{1}{2}$$

$$y = (c_1 z + c_2) e^{2z} + \frac{z^2}{2} e^{2z} - e^{z} - \frac{1}{2}$$
$$= e^{2z}\left[c_1 z + c_2 + \frac{z^2}{2}\right] - e^{z} - \frac{1}{2}$$
$$= u^2\left[c_1 z + c_2 + \frac{z^2}{2}\right] - u - \frac{1}{2}$$
$$= (x+1)^2\left[c_1 \log x) + c_2 + \frac{(\log x)^2}{2}\right] - x - 1 - \frac{1}{2}$$
$$= (x+1)^2\left[c_1 (\log x) + c_2 + \frac{(\log x)^2}{2}\right] - x - \frac{3}{2}$$

3, solve $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x^5$.

hw

<u>soln:</u>

<u>Gvn</u>, $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x^5 \longrightarrow (1)$

$u = x+a \quad \Rightarrow \quad x = u - a$

$\frac{du}{dx} = 1 \;, \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{dx}\left(\frac{dy}{du}\right) = \frac{d}{du}\left(\frac{dy}{du}\right) \cdot \frac{du}{dx}$$
$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}$$

$(1) \Rightarrow u^2 \frac{d^2y}{dx^2} - 4u\frac{dy}{dx} + 6y = (u-a)^5$

$(u^2D^2 - 4uD + 6)y = (u-a)^5 \longrightarrow (2)$

$u = e^z \;;\; z = \log x \;;\; \theta = \frac{d}{dx}$

$u^2D^2 = \theta(\theta - 1) \qquad uD = \theta$

$(\theta(\theta-1) - 4\theta + 6)y = (e^z - a)^5$

$(\theta^2 - \theta - 4\theta + 6)y = (e^z - a)^5$

$(\theta^2 - \theta - 4\theta + 6)y = e^{5z} - 2ae^{4z} + 10e^{3z}a^2 - 10e^{2z}a^3 + 5e^{z}a^4 - a$

$(\theta^2 - 5\theta + 6)y = e^{5z} - 2ae^{4z} + 10e^{3z}a^2 - 10e^{2z}a^3 + 5e^{z}a^4 - a^5$

To find CF

$$(\theta^2-5\theta+6)y=0$$

$$m^2-5m+6=0$$

$$(m-2)(m-3)=0$$

$$m=2,3$$

$$CF = c_1e^{2z}+c_2e^{3z}$$

To find PI

$$PI = \frac{1}{\theta^2-5\theta+6}\left[e^{5z}-2ae^{4z}+10e^{3z}a^2-10e^{2z}+5e^z a^4-a^5\right]$$

$$= \frac{e^{5z}}{6} - 2ae^{4z}+10a^2ze^{3z}+z\,10a^3e^{2z}+\frac{5a^4e^z}{2}-\frac{a^5}{6}$$

$$PI = \frac{2u^5-12au^4+30a^4u-2a^5}{12}+10a^2\log x\,u^2(u+a)$$

$$y = c_1u^2+c_2u^3+\frac{2u^5-12au^4+30a^4u-2a^5}{12}+10a^2\log x\,u^2(u+a)$$

4) solve $(1+2x)^2\cdot\frac{d^2y}{dx^2}+(1+2x)\frac{dy}{dx}+y=8(1+2x)^2$

Soln:

$$(1+2x)^2\cdot\frac{d^2y}{dx^2}+(1+2x)\frac{dy}{dx}+y=8(1+2x)^2 \longrightarrow (1)$$

$$u=1+2x$$

$$\frac{du}{dx}=2$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=2\cdot\frac{dy}{du}$$

$$=\frac{d}{dx}\left(\frac{dy}{du}\right)^2=\frac{d}{du}\left(2\frac{dy}{du}\right)\frac{du}{dx}$$

$$\frac{d^2y}{dx^2}=2\cdot\frac{d^2y}{du^2}\cdot 2$$

$$=2^2\cdot\frac{d^2y}{du^2}$$

$$(1) \Rightarrow u^2\cdot\frac{d^2y}{dx^2}+u\frac{dy}{dx}+y=8u^2$$

$$(u^2D^2+uD+1)y=8u^2$$

Put $u=e^z$, $z=\log x$, $\theta=\frac{d}{dz}$ $uD=\theta$, $u^2D^2=\theta(\theta+$

$$(\theta(\theta-1)+\theta+1)y=8e^{2z}$$

$(D^2 - D + D + 1)y = 8e^{2x}$

$(D^2+1)y = 8e^{2x}$

To find CF

The AE is $m^2+1 = 0$

$m^2 = -1$

$m = \pm i$

$CF = c_1 \cos x + c_2 \sin x$

To find P.I

$PI = \frac{1}{D^2+1} 8e^{2x}$

$= \frac{1}{4+1} 8e^{2x} \quad (D=2)$

$= \frac{8e^{2x}}{5}$

$PI = \frac{8}{5}e^{2x}$

$y = c_1 \cos x + c_2 \sin x + \frac{8}{5}e^{2x}$

$= c_1 \cos \log x + c_2 \sin \log x + \frac{8}{5}u^2$

$y = c_1 \cos \log x + c_2 \sin \log x + \frac{8}{5}(1+2x)^2$

SIMULTANEOUS LINEAR DIFFERENTIAL EQNS :

Pg 34 EG: 1

solve $\frac{dx}{dt} + 2x - 3y = t \rightarrow (1)$

$\frac{dy}{dt} - 3x + 2y = e^{2t} \rightarrow (2)$

soln:

Let D denoted by $\frac{d}{dt}$

$Dx + 2x - 3y = t$ $\qquad$ $Dy - 3x + 2y = e^{2t}$

$(D+2)x - 3y = t \longrightarrow (1)$ $\qquad$ $(D+2)y - 3x = e^{2t} \longrightarrow (2)$

Operating on (1) by $(D+2)$

$(D+2)^2 x - 3(D+2)y = (D+2)t \longrightarrow (3)$ $\qquad$ for cancelling (1) term

Mul x (2) by (3),

$-9x + 3(D+2)y = 3e^{2t} \longrightarrow$ (4)

solving (3) & (4)

$$(D+2)^2 x - 3(D+2)y = (D+2)t$$
$$-9x + 3(D+2)y = 3e^{2t}$$

$$[(D+2)^2 - 9]x = 3e^{2t} + (D+2)t$$

$(a+b)^2 = a^2+b^2+2ab$

$$[D^2+4+4D-9]x = 3e^{2t} + D(t) + 2t$$

$$D^2x + 4x + 4Dx - 9x = 3e^{2t} + 1 + 2t$$

$$[D^2+4D-5]x = 3e^{2t} + 1 + 2t \longrightarrow (5)$$

this is linear eqn with constant coefficient.

To find CF

$$(D^2+4D-5)x = 0$$

The AE is $m^2+4m-5 = 0$

$$m^2 - m + 5m - 5 = 0$$
$$m(m-1) + 5(m-1) = 0$$
$$(m-1)(m+5) = 0$$
$$m = 1, -5$$
$$CF = c_1e^t + c_2e^{-5t}$$

To find PI

$$PI = \frac{1}{D^2+4D-5}(1+2t+3e^{2t})$$

$$= \frac{1}{D^2+4D-5} + \frac{1}{D^2+4D-5}2t + 3\frac{1}{D^2+4t-5}e^{2t}$$

(-5 common)

$$= \frac{1}{-5\left(1-\frac{4}{5}D-\frac{D^2}{5}\right)} + 2\frac{1}{-5\left(1-\frac{4D}{5}-\frac{D^2}{5}\right)}t + \frac{3e^{2t}}{4+8-5}$$

$$= \frac{-1}{5}\left[1-\left(\frac{4}{5}D-\frac{D^2}{5}\right)\right]^{-1} - \frac{2}{5}\left[\left(1-\frac{4D}{5}-\frac{D^2}{5}\right)\right]^{-1}t + \frac{3e^{2t}}{7}$$

omit

$$= \frac{-1}{5}(1) - \frac{2}{5}\left[1+\frac{4D}{5}\right]t + \frac{3}{7}e^{2t}$$

$$= \frac{-1}{5} - \frac{2}{5}\left(1+\frac{4}{5}\right) + \frac{3e^{2t}}{7}$$

$= \frac{-1}{5} - \frac{2}{5}t - \frac{2}{25} + \frac{3}{7}e^{2t}$ $\frac{-1}{5} - \frac{2}{25}$

$= \frac{-5-8}{25} - \frac{2}{5}t + \frac{3}{7}e^{2t}$

$= \frac{-13}{25} - \frac{2t}{5} + \frac{3e^{2t}}{7}$

$x = c_1 e^{-5t} + c_2 e^{t} - \frac{13}{25} - \frac{2t}{5} + \frac{3e^{2t}}{7}$

$\frac{dx}{dt} = -5c_1 e^{-5t} + c_1 e^{t} - \frac{2}{5} + \frac{6}{7}e^{2t}.$

subs the values of x & $\frac{dx}{dt}$ in ①,

$-5c_1 e^{-5t} + c_2 e^{t} - \frac{2}{5} + \frac{6}{7}e^{2t} + 2\left(c_1 e^{-5t} + c_1 e^{t} - \frac{13}{25} - \frac{2t}{5} + \frac{3e^{2t}}{7}\right) - 3y = t$

$-3y = t + 5c_1 e^{-5t} - c_2 e^{t} + \frac{2}{5} - \frac{6e^{2t}}{7} - 2c_1 e^{-5t} - 2c_2 e^{t} + \frac{26}{25} + \frac{4t}{5} - \frac{6}{7}e^{2t}$

$-3y = \left(1+\frac{4}{5}\right)t + (5c_1 - 2c_1)e^{-5t} - (c_2 + 2c_2)e^{t} + \left(\frac{-6}{7} - \frac{6}{7}\right)e^{2t} + \frac{2(5)+26}{25}$

$-3y = \frac{9}{5}t + 3c_1 e^{-5t} - 3c_2 e^{t} - \frac{12}{7}e^{2t} + \frac{36}{25}$

$y = \frac{-3}{5}t + c_1 e^{-5t} - c_2 e^{t} - \frac{4}{7}e^{2t} + \frac{12}{25}$

$y = c_1 e^{t} - c_2 e^{-5t} - \frac{12}{25} - \frac{36}{5} + \frac{4e^{2t}}{7}.$

Exer ① solve: $3(1-D)x + 4y = 3t+1 \longrightarrow$ ①

$3(D+1)y + 2x = e^{t}$

Soln :- $3(1-D)x + 4y = 3t+1 \longrightarrow$ ①

$\times 3(D+1)$ $\times 4$ $3(D+1)y + 2x = e^{t} \longrightarrow$ ②

$3(1-D)x \cdot 3(D+1) + 4 \cdot 3(D+1)y = (3t+1)\,3(D+1) \rightarrow$ ③

$3(D+1)y \cdot 4 + 4 \cdot 2x = 4e^{t} \longrightarrow$ ④

$9(1-D)(D+1)x + 12(D+1)y = (3t+1)(3(D+1))$

$8x + 12(D+1)y = 4e^{t}$

$(-) \quad (-) \quad (-)$

$(9(1-D)(D+1) - 8)x = (3t+1)\cdot 3D + 3 - 4e^{t}$

$$(9(D+1-D^2-D)-8)x = (3t+1)\cdot 3D+3-4e^t$$

$$(9-9D^2-8)x = 9t(D)+3D+3-4e^t$$

$$(9-9D^2-8)x = (9t+3D+3)-4e^t$$ Doubt

$$(9-9D^2-8)x = 12-4e^t$$

$$(9D^2-1)x = 12-4e^t+9t$$

To find CF

$$CF = 9D^2-1=0$$

$$9m^2-1=0$$

$$9m^2=1$$

$$m^2=\frac{1}{9}$$

$$m=\pm\frac{1}{3}$$

$$CF = c_1e^{-\frac{1}{3}t}+c_2e^{\frac{1}{3}t}$$

To find PI

$$= \frac{1}{9D^2-1}(9t+12)-4e^t$$

$$= \frac{-4e^t}{1-9D^2}+\frac{9t}{1-9D^2}+\frac{12}{1-9D^2}$$

$$= \frac{-4e^t}{-8}+9\frac{t}{1-9D^2}+12\cdot\frac{1}{1-9D^2}$$

$D^2=1$

$$= \frac{e^t}{2}+9\left(\frac{1}{1-9D^2}\right)t+12$$ common $(1-9D^2)$

$$= \frac{e^t}{2}+9(1-9D^2)^{-1}t+12$$

$$= \frac{e^t}{2}+9(1+9D^2)t+12$$

$$= \frac{e^t}{2}+9[1+9D^2(t)]+12$$

$$= \frac{e^t}{2}+9t+\underbrace{81D^2(t)}_{0}+12$$

$$= \frac{e^t}{2}+9t+12$$

$$PI = e^{t/2}+9t+12$$

$$x = c_1e^{1/3t}+c_2e^{-1/3t}+12+9t+e^t/2.$$

12. solve $\frac{d^2x}{dt^2} - 3x - 4y = 0$ & $\frac{d^2y}{dt^2} + x + y = 0$

soln

$$\frac{d^2x}{dt^2} - 3x - 4y = 0 \longrightarrow (1)$$

$$\frac{d^2y}{dt^2} + x + y = 0 \longrightarrow (2)$$

Denote $D = \frac{d}{dt}$

~~$(D^2-3)x -$~~

~~$(D^2$~~ $(D^2-3)x - 4y = 0 \longrightarrow (1)$ $\quad (D^2+1)x$

$(D^2+1)y + x = 0 \longrightarrow (2)$ $\quad (-4)$

operating on (1) by (D^2+1)

$$(D^2+1)(D^2-3)x - 4y(D^2+1) = 0$$

$\times (-4)$ on (2)

$$-4(D^2+1)y - 4x = 0$$

$$(D^2+1)(D^2-1)x - 4y(D^2+1) = 0$$

$$\underset{(+)}{-4x} \quad \underset{(+)}{-4y}(D^2+1)y = 0$$

$$((D^2+1)(D^2-3)+4)x = 0$$

$$(D^4 - 3D^2 + D^2 - 3 + 4)x = 0$$

$$(D^4 - 2D^2 + 1)x = 0$$

$$(D^2-1)^2 x = 0$$

To find CF

The AE is $(m^2-1)^2 = 0 \rightarrow$ no need PI

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$ (twice)

The roots are real
1 is repeated twice

is repeated

$CF = e^{t}(c_1+c_2t) + e^{-t}(c_3+c_4t)$ finish

$x = e^{t}(c_1+c_2t) + e^{-t}(c_3+c_4t) \longrightarrow (3)$

$\frac{dx}{dt} = e^{t}(c_1+c_2\cancel{t}) - e^{-t}(c_3+c_4t)$

$\frac{d^2x}{dt^2} = e^{t}(c_1+c_2t) + e^{-t}(c_3+c_4t)$

sub eqn (1) $e^{t}(c_1+c_2t)+e^{-t}(c_3+c_4t) - 3(e^{t}(c_1+c_2t)+e^{-t}(c_3+c_4t)$
$-4y = 0$

$4y = e^{t}(c_1+c_2t) + e^{-t}(c_3+c_4t) - 3e^{t}(c_1+c_2t) - 3e^{-t}(c_3+c_4t)$

$\frac{dx}{dt} = e^{t}(c_2) + (c_1+c_2t)e^{t} + e^{-t}(c_4) - e^{-t}(c_3+c_4)t$

$= e^{t}(c_1+c_2+c_2t) + e^{-t}(c_3-c_4t-c_4)$

$= e^{t}(c_1+(1+t)c_2) + e^{-t}(c_4(1-t)-c_3)$

$\frac{d^2x}{dt^2} = e^{t}(c_2) + (c_1+(1+t)c_2)\cdot e^{t} + e^{-t}(-c_4) - e^{t}$
$(c_4(1-t)-c_4)$

$= e^{t}(c_2+c_1+c(1+t)c_2) + e^{-t}(-c_4-c_4(1-t)+c_3)$

$= e^{t}(c_2(2+t)+c_1) + e^{-t}(c_3-c_4(t-2))$

$= e^{t}(2c_2+c_2t+c_1) + e^{-t}(c_3-2c_4+c_4t)$

(1) $\Rightarrow$ $e^{t}(2c_2+c_2t+c_1) + e^{-t}(c_3-2c_4+c_4t) - 3(e^{t}(c_1+c_2t)$
$+e^{-t}(c_3+c_4t) - 4y = 0$

$4y = e^{t}(2c_2+c_2t+c_1) + e^{-t}(c_3-2c_4+c_4t) - 3e^{t}(c_1+c_2t)$
$-3e^{-t}(c_3+c_4t)$

$4y = e^{t}(2c_2+c_2t+c_1-3c_1-3c_2t) + e^{-t}(c_3-2c_4+c_4t$
$-3c_3-3c_4t)$

$4y = e^{t}(-2c_1+2c_2-2c_2t) + e^{-t}(-2c_3-2c_4-2c_4t)$

2) $(D+5)x+y=e^t$; $(D+3)y-x=e^{2t}$

<u>soln</u> :-

$$(D+5)x+y=e^t \longrightarrow ①$$

$$(D+3)y-x=e^{2t} \longrightarrow ②$$

Operating on ① by $(D+3)$

$$(D+5)(D+3)x + (D+3)y = (D+3)e^t$$

$$\underset{(+)}{-x} \underset{(-)}{+} (D+3)y = \underset{(-)}{e^{2t}}$$

$$((D+5)(D+3)+1)x = (D+3)e^t - e^{2t}$$

$$((D^2+3D+5D+15)+1)x = De^t+3e^t-e^{2t}$$

$$((D^2+8D+15)+1)x = De^t+3e^t-e^{2t}$$

$$= e^t+3e^t-e^{2t}$$

$$((D^2+8D+15)+1)x = 4e^t-e^{2t}$$

This is linear eqn with constant coefficient

<u>To find CF</u>

The AE is $m^2+8m+16=0$

$$(m+4)^2=0$$

$$m+4=0$$

$$m=-4 \text{ (twice)}$$

$$CF = e^{-4t}(c_1+c_2t)$$

<u>To find PI</u>

$$PI = \frac{1}{D^2+8D+16} 4e^t - e^{2t}$$

$$= \frac{1}{1+8D+16} 4e^t - \frac{1}{4+16+16} e^{2t}$$

$$= \frac{4e^t}{25} - \frac{e^{2t}}{36}$$

$$x = e^{-4t}(c_1+c_2t) + \frac{4e^t}{25} - \frac{e^{2t}}{36} \longrightarrow ③$$

$$\frac{dx}{dt} = e^{-4t}(c_2) + (c_1+c_2t) - 4e^{-4t} + \frac{4e^t}{25} - \frac{2e^{-2t}}{18}$$

$$= e^{-4t}(c_2 - 4c_1e^{-4t} - 4e^{-4t}c_2t + \frac{4}{25}e^t - \frac{e^{2t}}{18}$$

$$\frac{dx}{dt} = e^{-4t}c_2(1-4t) - 4c_1e^{-4t} + \frac{4e^t}{25} - \frac{e^{2t}}{18} \longrightarrow (6)$$

(5) & (6) subs in (1),

$$e^{-4t}c_2(1-4t) - 4c_1 e^{-4t} + \frac{4e^t}{25} - \frac{e^{2t}}{18} + 5\left(e^{-4}(c_1 5c_2) + \frac{4e^t}{25} - \frac{e^{2t}}{36} + y = e^t\right.$$

$$e^{-4t}c_2 - 4c_2e^{-4t}t - 4c_1e^{-4t} + \frac{4e^t}{25} - \frac{e^{2t}}{18} + 5e^{-4t}c_1 + 5e^{-4t}c$$

$$+ \frac{4}{5}e^t - \frac{5e^{2t}}{36} + y = e^t$$

$$e^{-4t}c_2t + e^{-4t}c_1 + \frac{4e^t}{25} + \frac{4}{5}e^t - e^t - \frac{5e^{2t}}{36} - \frac{e^{2t}}{18} + e^{-4t}c_2 +$$

$$e^{-4t}c_2t + e^{-4t}c_1 + \frac{4e^t + 20e^t - 25e^t}{25} - \frac{90e^{2t} - 36e^{2t}}{648} + e^{-4t}c_2 + y$$

$$e^{-4t}c_2(t+1) + e^{-4t}c_1 - \frac{e^t}{25} - \frac{126e^{2t}}{648} + y = 0$$

$$e^{-4t}c_2(t+1) + e^{-4t}c_1 - \frac{e^t}{25} - \frac{7}{36}e^{2t} + y = 0$$

$$y = \frac{e^t}{25} + \frac{7}{36}e^{2t} - e^{-4t}c_1 - e^{-4t}c_2(t+1)$$

SIMULTANEOUS EQNS OF THE FIRST ORDER AND FIRST DEGREE

(i) solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \rightarrow (1)$

where P, Q, R are func of x, y, z

$\phi(u, v) = 0$ is soln of (1).

METHOD OF GROUPING :

Ex 1

solve the eqn $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$

soln :

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

taking 1st & 2nd ratios

$$\frac{dx}{yz} = \frac{dy}{xz}$$

$$\frac{dx}{y} = \frac{dy}{x}$$

$$xdx = ydy$$

Integrating

$$\int xdx = \int ydx$$

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

$$\frac{x^2}{2} - \frac{y^2}{2} = c_1$$

Taking 2nd & 3rd ratios

$$\frac{dy}{xz} = \frac{dz}{xy}$$

$$\frac{dy}{z} = \frac{dz}{y}$$

$y\,dy = z\,dz$

Integrating

$$\int y\,dy = \int z\,dz$$

$$\frac{y^2}{2} = \frac{z^2}{2} + c_2$$

$$\frac{y^2}{2} - \frac{z^2}{2} = c_2$$

soln is $\phi(u, v) = 0$

$$\phi\left(\frac{x^2}{2} - \frac{y^2}{2}, \frac{y^2}{2} - \frac{z^2}{2}\right) = 0$$

Ex : 4

solve : $\dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{x(yz - 2x)}$

<u>soln</u> :-

$$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{x(yz - 2x)}$$

taking 1st & 2nd Ratios

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log\left(\frac{x}{y}\right) = \log c_1$$

$$\frac{x}{y} = c_1 \rightarrow (1)$$

$$\Rightarrow y = \frac{x}{c_1}$$

Taking 1st & last Ratio

$$\frac{dx}{xy} = \frac{dz}{x(yz-2x)}$$

$$\frac{dx}{y} = \frac{dz}{yz-2x} \quad (\because \text{By } 1)$$

$$\frac{dx}{\frac{x}{c_1}} = \frac{dz}{\frac{xz}{c_1} - 2x}$$

$$\frac{dx}{\frac{x}{c_1}} = \frac{dz}{\frac{x}{c_1}(z-2c_1)}$$

$$\frac{dx}{1} = \frac{dz}{z-2c_1}$$

$\int$ we get

$$\int dx = \int \frac{dz}{z-2c_1}$$

$$x = \log(z-2c_1) + \log c_2$$

$$x = \log\left(z - 2\frac{x}{y}\right) \cdot c_2$$

$$e^x = e^{\log\left(z-\frac{2x}{y}\right)} c_2$$

$$e^x = \left(z - \frac{2x}{y}\right) \cdot c_2$$

$$e^x = \left(\frac{zy-2x}{y}\right) c_2$$

$$ye^x = (zy-2x)\, c_2$$

$$c_2 = \frac{ye^x}{yz-2x} \rightarrow 2$$

soln is $\phi(u,v) = 0$

ϕ is arbitrary,

$$\phi\left(\frac{x}{y}, \frac{ye^x}{yz-2x}\right) = 0$$

MULTIPLIER METHOD :

we have $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

where P, Q, R are func of x, y, z

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

$$= \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$$

where l, m, n & l', m', n' are multipliers are the two sets of multipliers, not necessary constants.

If l, m, n are chosen that,

$$lP + mQ + nR = 0, \text{ then}$$

$$l\,dx + m\,dy + n\,dz = 0.$$

PROBLM

Ex : 2

solve the eqn : $\frac{dx}{-y^2 - z^2} = \frac{dy}{xy} = \frac{dz}{xz}$

The general soln,

$\phi(u, v) = 0$ which ϕ is arbitrary of n

Taking 2nd & 3rd ratios,

$$\frac{dy}{xy} = \frac{dz}{xz}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log c$$

$$\log y - \log z = \log c$$

$$\log\left(\frac{y}{z}\right) = \log c$$

$$\boxed{\frac{y}{z} = c_1} \rightarrow (1)$$

$$\frac{dx}{-y^2-z^2} = \frac{dy}{xy} = \frac{dz}{xz}$$

using the multipliers x, y, z

$$lP + mQ + nR = x(-y^2-z^2) + yxy + zxz$$

$$= -xy^2 - xz^2 + xy^2 + xz^2$$

$$= 0$$

$$\frac{dx}{-y^2-z^2} = \frac{dy}{xy} = \frac{dz}{xz} = \frac{xdx + ydy + zdz}{x(-y^2-z^2) + xy^2 + xz^2}$$

$$= \frac{xdx + ydy + zdz}{0}$$

$\therefore xdx + ydy + zdz = 0$

Integrating, we get,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$\frac{x^2+y^2+z^2}{2} = c$$

$$x^2+y^2+z^2 = 2c$$

$$\boxed{x^2+y^2+z^2 = c_2} \quad \text{where } c_2 = 2c$$

gn soln is $\phi\left(\frac{y}{z}, x^2+y^2+z^2\right) = 0$ where ϕ is arbitary.

Eg: 7

solve $\frac{dx}{mz-ny} = \frac{dy}{\cancel{mx-lz}\; nx-lz} = \frac{dz}{ly-mx}$

soln:

gn $\frac{dx}{mz-ny} = \frac{dy}{\cancel{mx-nz}\; nx-lz} = \frac{dz}{ly-mx}$

Using multipleirs x, y, z

$$lP + mQ + nR = x(mz-ny) + y(nx-lz) + z(ly-mx)$$

$$= mxz - nxy + nxy - lyz + lyz - mxz$$

$$= 0$$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{xdy + ydy + zdz}{x(mz-ny)+y(nz-lz)+z(ly-mx)}$$

$$= \frac{xdx+ydy+zdz}{0}$$

$\therefore xdx + ydy + zdz = 0.$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \cancel{0}\ c$$

$$\frac{x^2+y^2+z^2}{2} = \cancel{0}\ c$$

$$x^2+y^2+z^2 = c$$

$$x^2+y^2+z^2 = 2c$$

$$x^2+y^2+z^2 = c_1 \quad (c_1 = 2c)$$

Using the multiplier m, n & l

$$lP + mQ + nR = l(mz-ny) + m(nx-lz) + n(ly-mx)$$

$$= lmz - lny + mnx - mlz + nly - mnx$$

$$= 0$$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{xdx + ydy + ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}$$

$$= \frac{xdx+ydy+zdz}{0}$$

$\therefore ldx + mdy + ndz = 0$

Integrating,

$$lx + my + nz = c_2$$

Gen soln is $\phi(u, v) = 0$ where ϕ is arbitrary

soln is $\phi(x^2+y^2+z^2, lx+my+nz) = 0$

HOMEWORK

1. $\dfrac{dx}{y+zx} = \dfrac{dy}{-x-yz} = \dfrac{dz}{x^2-y^2}$

soln:

Using multiplier x, y, z

$$lP + mQ + nR = x(y+zx) + y(-x-yz) + z(x^2-y^2)$$

$$= xy + x^2z - xy - y^2z + x^2z - y^2z$$

$$= 0$$

$$\frac{dx}{y+zx} = \frac{dy}{-x-yz} = \frac{dz}{x^2-y^2} = \frac{xdx + ydy - zdz}{x(y+zx) + y(-x-yz) - z(x^2-y^2)}$$

$$= \frac{xdx + ydy - zdz}{0}$$

$$\therefore xdx + ydy - zdz = 0$$

Integrating, $\dfrac{x^2}{2} + \dfrac{y^2}{2} - \dfrac{z^2}{2} = c_1$

$$x^2 + y^2 - z^2 = c_1$$

$$\frac{ydx}{y(y+zx)} + \frac{xdy}{x(-x-yz)} = \frac{dz}{y^2-x^2}$$

$$\frac{ydx + xdy}{y^2 + xyz - x^2 - xyz} = \frac{dz}{y^2-x^2}$$

$$\frac{ydx + xdy}{y^2 - x^2} = \frac{dz}{y^2-x^2}$$

$$ydx + xdy = dz$$

$$d(xy) = dz$$

Integrating, $xy = z + c_2$

$$xy - z = c_2$$

Gn soln is $\phi(u, v) = 0$

$$\phi(x^2+y^2-z^2,\ xy - z) = 0$$

2. solve $\frac{dx}{xz} = \frac{dy}{yz} = \frac{2dz}{(x+y)^2}$

<u>soln:</u>

taking 1 & 2 Ratios

$$\frac{dx}{xz} = \frac{dy}{yz}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\log x = \log y + \log c$$

$$\log x - \log y = \log c$$

$$\frac{x}{y} = c_1$$

$$\boxed{x = c_1 y}$$

Taking I & II Ratio

$$\frac{dx + dy}{z(x+y)} = \frac{2dz}{(x+y)^2}$$

$$\frac{dx+dy}{z} = \frac{2dz}{x+y}$$

$$(x+y)\,dx + dy = 2z\,dz$$

$$(x+y)\,d(x+y) = 2z\,dz$$

$$\frac{(x+y)^2}{2} = z^2 + c$$

$$(x+y)^2 = 2z^2 + c_2$$

$$(x+y)^2 - 2z^2 = c_2$$

Gn soln is $\phi\left(x/y,\ (x+y)^2 - 2z^2\right) = 0$

solve $\frac{y\,dx}{y^2+z^2-xz} = \frac{dy}{2xy} = \frac{dz}{2xz}$

<u>soln:</u>

taking II & III Ratios

$$\frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{dy}{2y} = \frac{dz}{2z}$$

$$\frac{1}{2}\log y = \frac{1}{2}\log z + \log c$$

$$\log y^{1/2} = \log z^{1/2} + \log c$$

$$\log y^{1/2} - \log z^{1/2} = \log c$$

$$\frac{\sqrt{y}}{\sqrt{z}} = c_1$$

$$\left(\frac{y}{z}\right)^{1/2} = c_1$$

$$\frac{y}{z} = c_1$$

Multiplying, $3x, -y, -z$

$$\frac{3xdx - ydy - zdz}{x(y^2+z^2-3x^2)} = \frac{dz}{2xz}$$

$$\frac{3xdy - ydy - zdx}{y^2+z^2-3x^2} = \frac{dz}{2z}$$

Multiply by -2,

$$\frac{2ydy + 2zdz - 6xdx}{y^2+z^2-3x^2} = -\frac{\cancel{2}dz}{\cancel{2}z}$$

$$\frac{d(y^2+z^2-3x^2)}{y^2+z^2-3x^2} = -\frac{dz}{z}$$

$$\log(y^2+z^2-3x^2) = -\log z + \log c_2$$

$$(y^2+z^2-3x^2)z = c_2$$

gen soln $\phi\left(\frac{y}{z}, (y^2+z^2-3x^2)z = c_2\right)$

4. $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

soln :-

$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

$x(y-z) + y(z-x) + z(x-y) = 0$

$xy - xz + yz - xy + xz - yz = 0 \longrightarrow (1)$

$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{dx+dy+dz}{0}$

Then $dx + dy + dz = 0$

Integrating,

$x + y + z = c.$

Taking 1^{st} & 2 ratios,

$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} \Rightarrow \frac{ydx + xdy}{xy^2 - xyz + xyz - x^2y}$

$= \frac{ydx + xdy}{xy^2 - x^2y}$

$= \frac{ydx + xdy}{xy(y-x)}$

$\frac{ydx + xdy}{-xy(x-y)} = \frac{dz}{z(x-y)}$

$\frac{d(xy)}{-xy} = \frac{dz}{z}$

$-\log xy = \log z + \log c_2$

$\log c_2 = \log(xyz)$

$c_2 = xyz$

soln is $\phi(x+y+z, xyz)$

UNIT - III

LINEAR DIFF EQNS OF THE 2^{nd} ORDER USING VARIABLE COEFFICIENT :

The general eqn is,

$$\frac{d^2y}{dx^2} + P \cdot \frac{dy}{dx} + Qy = R \longrightarrow (1)$$

where P, Q, R are func of x - alone.

METHOD OF REDUCTION OF ORDER :

Let $$\frac{d^2y}{dx^2} + P \cdot \frac{dy}{dx} + Qy = R \longrightarrow (1)$$

where R = 0,

$$\frac{d^2y}{dx^2} + P \cdot \frac{dy}{dx} + Qy = 0 \longrightarrow (2)$$

Let $y = uv$ be the soln of (1)

Let $y = u$ be the known soln of (2),

we shall determine v.

To find the soln of (2), we have,

(1) If $1+P+Q=0$ then $y = e^{x}$ is a soln of (2)

(2) If $1-P+Q=0$ then $y = e^{-x}$ is a soln of (2)

(3) If $a^2+aP+Q=0$ then $y = e^{ax}$ is a soln of (2)

(4) If $P+Qx=0$ then $y = x$ is a soln of (2)

(5) If $2+2Px+Qx^2=0$ then $y = x^2$ is a soln of (2)

PROBLEM

EG : 1 $\boxed{\text{Pg : 146}}$

solve : $\frac{x d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$

soln :

Gn that $x \cdot \frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$

Dividing throughout x,

$$\frac{d^2y}{dx^2} - \frac{(2x-1)}{x}\frac{dy}{dx} + \frac{(x-1)}{x}y = \frac{e^x}{x} \longrightarrow (1)$$

this is of the form,

$$\frac{d^2y}{dx^2} + P \cdot \frac{dy}{dx} + Qy = R$$

here,

$$P = -\frac{(2x-1)}{x}$$

$$Q = \frac{x-1}{x}$$

$$R = \frac{e^x}{x}$$

Let $y = uv$ be the soln of (1),

Let $\frac{d^2y}{dx^2} + \frac{-(2x-1)}{x}\frac{dy}{dx} + \frac{(x-1)}{x}y = 0 \longrightarrow (2)$

To find u :- let $y = u$

$$1 + P + Q = 1 - \frac{(2x-1)}{x} + \frac{x-1}{x}$$

(e^x so $1+P$ form)

$$= \frac{x - 2x + 1 + x - 1}{x}$$

$$1 + P + Q = 0$$

$\therefore \boxed{y = e^x}$ is soln of (2)

Then $y = e^x v \longrightarrow$ ③

$$\frac{dy}{dx} = e^x \cdot \frac{dv}{dx} + e^x \cdot v \longrightarrow ④$$

$$\frac{d^2y}{dx^2} = e^x \cdot \frac{d^2v}{dx^2} + e^x \cdot \frac{dv}{dx} + e^x \cdot \frac{dv}{dx} + e^x \cdot v$$

$$= e^x \cdot \frac{d^2v}{dx^2} + 2e^x \frac{dv}{dx} + e^x v \longrightarrow ⑤$$

sub ③, ④, ⑤ in ①,

$$e^x \cdot \frac{d^2v}{dx^2} + 2e^x \cdot \frac{dv}{dx} + e^x v - \frac{(2x-1)}{x}\left(e^x \cdot \frac{dv}{dx} + e^x \cdot v\right) + \frac{(x-1)}{x} e^x \cdot v = \frac{e^x}{x}.$$

Cancelling e^x throughout, we get,

$$\frac{d^2v}{dx^2} + 2 \cdot \frac{dv}{dx} + v - \frac{(2x-1)}{x}\left(\frac{dv}{dx} + v\right) + \frac{(x-1)}{x} v = \frac{1}{x}$$

$$\frac{d^2v}{dx^2} + 2 \cdot \frac{dv}{dx} + v - \frac{(2x-1)}{x}\frac{dv}{dx} - \frac{(2x-1)}{x} v + \frac{(x-1)}{x} v = \frac{1}{x}.$$

$$\frac{d^2v}{dx^2} + \left(\frac{2x-2x+1}{x}\right)\frac{dv}{dx} + \left(\frac{x-2x+1+x-1}{2}\right) v = 1/x$$

$$\frac{d^2v}{dx^2} + \frac{1}{x} \cdot \frac{dv}{dx} = \frac{1}{x} \longrightarrow ✱$$

Let $\boxed{\frac{dv}{dx} = P} \longrightarrow$ ⑥

Then, $\boxed{\frac{d^2v}{dx^2} = \frac{dP}{dx}}$

✱ becomes, $\frac{dP}{dx} + \frac{1}{x}P = \frac{1}{x}$

This is linear eqn

$$P_1 = \frac{1}{x}, \quad Q_1 = \cancel{\frac{1}{x}} \ 1/x$$

$$P \cdot e^{\int P_1 dx} = \int Q \cdot e^{\int P_1 dx} \, dx + c$$

$$e^{\int P_1 dx} = e^{\int \frac{1}{x} dx}$$

$= e^{\log x}$

$e^{\int P_1 dx} = x$

$P.x = \int \frac{1}{x} \cdot x \cdot dx + c$

$P.x = \int dx + c$

$Px = x + c$

$Px - x = c$

$x(P-1) = c$

$P - 1 = \frac{c}{x}$

$\boxed{P = 1 + \frac{c}{x}}$

$\frac{dv}{dx} = 1 + \frac{c}{x}$ [from 6]

Integrating,

$\int dv = \int \left(1 + \frac{c}{x}\right) dx$ $\quad \int dx = x$

$\boxed{v = x + c \log x + c_1}$

Hence, $\boxed{y = e^x . V}$ $\quad u = e^x.$

$y = e^x (x + c_1 + c \log x)$

xxIII

P2 156 5. solve $(x \sin x + \cos x) \frac{d^2y}{dx^2} - x \cos x \frac{dy}{dx} + y \cos x = 0$

soln:-

Gn that $(x \sin x + \cos x) \frac{d^2y}{dx^2} - x \cos x \frac{dy}{dx} + y \cos x = 0$

Dividing $(x \sin x + \cos x)$ throughout,

$\frac{d^2y}{dx^2} - \frac{x \cos x}{x \sin x + \cos x} \frac{dy}{dx} + \frac{\cos x}{x \sin x + \cos x} y = 0 \rightarrow (1)$

Let $y = uv$ be soln of (1),

$P = - \frac{x \cos x}{x \sin x + \cos x}$

$Q = \frac{\cos x}{x \sin x + \cos x}$

$R = 0$

$1+P+Q \neq 0, \quad 1-P+Q \neq 0, \quad a^2+ap+Q \neq 0$

$$P+Qx = \frac{-x\cos x}{x\sin x+\cos x} + \frac{\cos x}{x\sin x+\cos x}\cdot x$$

$$P+Qx = 0$$

$y = x$ is soln of $\rightarrow$ ①

Hence $\boxed{y = xv} \rightarrow$ ② $\quad \boxed{u=x}$

$$\frac{dy}{dx} = x\cdot\frac{dv}{dx} + v \rightarrow ③$$

$$\frac{d^2y}{dx^2} = x\cdot\frac{d^2v}{dx^2} + \frac{dv}{dx} + \frac{dv}{dx}$$

$$= x\cdot\frac{d^2v}{dx^2} + 2\cdot\frac{dv}{dx} \rightarrow ④$$

sub ②, ③, ④ in ①

$$x\cdot\frac{d^2v}{dx^2} + 2\cdot\frac{dv}{dx} - \frac{x\cos x}{x\sin x+\cos x}\left(x\cdot\frac{dv}{dx}+v\right) + \frac{\cos x}{x\sin x+\cos x}(xv) = 0$$

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \frac{x^2\cos x}{x\sin x+\cos x}\frac{dv}{dx} - v\frac{x\cos x}{x\sin x+\cos x} + \frac{\cos x}{x\sin x+\cos x}(xv) = 0$$

$$x\frac{d^2v}{dx^2} + \frac{2(x\sin x+\cos x) - x^2\cos x}{x\sin x+\cos x}\frac{dv}{dx} + \left(\frac{x\cos x}{x\sin x+\cos x} - \frac{x\cos x}{x\sin x+\cos x}\right)v = 0$$

$$x\frac{d^2v}{dx^2} + \left(\frac{2(x\sin x+\cos x) - x^2\cos x}{x\sin x+\cos x}\right)\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + \left(\frac{2(x\sin x+\cos x - x^2\cos x)}{x^2\sin x+x\cos x}\right)\frac{dv}{dx} = 0 \longrightarrow (*)$$

Let $\frac{dv}{dx} = P$

$$\frac{d^2v}{dx^2} = \frac{dP}{dx}$$

Subs in (*)

$$\frac{dP}{dx} + \left(\frac{2(x\sin x + 2\cos x - x^2\cos x)}{x^2\sin x + x\cos x}\right) P = 0$$

$$R = \frac{2x\sin x + 2\cos x - x^2\cos x}{x^2\sin x + x\cos x} \qquad Q = 0$$

$$IF = e^{\int \frac{2x\sin x + 2\cos x - x^2\cos x}{x^2\sin x + x\cos x}\,dx} \qquad e^{\int R\,dx}$$

$$= e^{\int x\sin x + \cos x\left(2 - x^2\cos x\right)}$$

$$= e^{\int \frac{2(x\sin x + \cos x) - x^2\cos x}{x^2\sin x + x\cos x}}$$

$$= e^{\int \frac{2(x\sin x + \cos x)}{x(x\sin x + \cos x)} - \frac{x^2\cos x}{x^2(\sin x + \cos x)}}$$

$$= e^{\int \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right) dx} \quad x\cos x + \sin x - \sin x$$

$$= e^{2\log x - \log(x\sin x + \cos x)}$$

$$= e^{\log x^2 - \log(x\sin x + \cos x)}$$

$$= \log \frac{x^2}{x\sin x + \cos x}$$

$$IF = \frac{x^2}{x\sin x + \cos x}$$

$$P \cdot \frac{x^2}{x\sin x + \cos x} = C$$

$$P = \frac{C(x\sin x + \cos x)}{x^2}$$

$$\frac{dv}{dx} = \frac{C(x\sin x + \cos x)}{x^2}$$

$$\int dv = \int \frac{(x\sin x + \cos x)}{x^2}\,dx$$

$$v = \int \frac{x\sin x + \cos x}{x^2}\,dx$$

$$v = \int c\frac{x\sin x}{x^2} + \frac{\cos x}{x^2}$$

$$= \int c\frac{\sin x}{x} + \frac{\cos x}{x^2} \longrightarrow (*)$$

Let $\int c\,\frac{\sin x}{x}$

$u = \frac{1}{x} \qquad dv = \sin x$

$du = \frac{-1}{x^2} \qquad v = -\cos x$

$$= \frac{-1}{x}\cos x + \int \cos x \cdot \frac{1}{x^2}$$

(*) becomes,

$$v = -\left[c\,\frac{1}{x}\cos x - \int\frac{\cos x}{x^2} + \int\frac{\cos x}{x^2} + c_2\right]$$

$$= -\frac{c}{x}\cos x + c_2$$

$$v = -\frac{c\cos x + c_2 x}{x}$$

Hence $y = uv$ $\qquad u = x$

$$= x - \frac{(c\cos x + c_2 x)}{x}$$

$y = c_2 x - c\cos x$ $\qquad$ let $c_1 = -c$

$$\boxed{y = c_2 x + c_1\cos x}$$

REDUCTION TO THE NORMAL FORM

(OR)

REMOVING THE 1^{st} DERIVATIVE

Consider,

$$\frac{d^2y}{dx^2} + P.\frac{dy}{dx} + Qy = R \longrightarrow (1)$$

choose, $u = e^{-\frac{1}{2}\int P dx}$.

(1) Reduces to,

$$\frac{d^2V}{dx^2} + Q_1 V = R_1 \longrightarrow (2)$$

where, $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{u}$ & $R_1 = \frac{R}{u}$.

The new eqn (2) in which the 1^{st} derivative, is absent is called " Removing the " ". Also (1) is said to be Reduced to the normal form (2).

PROBML :-

1. solve $y_2 - 4xy_1 + (4x^2-3)y = e^{x^2}$

soln :-

Gn. T. $y_2 - 4xy_1 + (4x^2-3)y = e^{x^2}$

This is of the form, $\frac{d^2y}{dx^2} + P.\frac{dy}{dx} + Qy = R$.

$P = -4x \quad Q = 4x^2-3 \quad R = e^{x^2}$.

Let $y = uv$ be the soln of (1)

Choose, $u = e^{-\frac{1}{2}\int P dx}$

$$= e^{2\int x dx}$$

$$= e^{2 \cdot \frac{x^2}{2}}$$

$$u = e^{x^2}$$

① Reduces to

$$\frac{d^2V}{dx^2} + Q_1 V = R_1 \longrightarrow ②$$

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$ & $R_1 = \frac{R}{u}$.

$$Q_1 = (4x^2-3) - \frac{1}{2}(-4) - \frac{(-4x)^2}{4}$$

$$= 4x^2 - 3 + 2 - \frac{16x^2}{4}$$

$$= 4x^2 - \frac{16x^2}{4} - 1 \Rightarrow -1$$

$$\boxed{Q = -1}$$

$$R_1 = \frac{R}{u}$$

$$= \frac{e^{x^2}}{e^{x^2}} = 1$$

$$\boxed{R_1 = 1}$$

② $\Rightarrow \frac{d^2V}{dx^2} - V = 1$

we have $V = CF + PI$

To find CF

$$(D^2-1)V = 0$$

The AE is $m^2 - 1 = 0$

$$m^2 = 1$$

$$m = \pm 1$$

Real & distinct

$$CF = c_1 e^{x} + c_2 e^{-x}$$

To find PI

$$PI = \frac{1}{D^2-1}(1)$$

$$PI = \frac{1}{-(1-D^2)}(1)$$

$$PI = -(1-D^2)^{-1}(1)$$

$$= -(1)$$

$$\boxed{PI = -1}$$

$v = c_1 e^{-x} + c_2 e^{x} - 1$

soln is $y = uv$

$y = e^{x^2}(c_1 e^{-x} + c_2 e^{x} - 1)$

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2) solve $x^2 \frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3 e^x$.

soln:-

Gn. T. $x^2 \frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3 e^x$.

$$\frac{d^2y}{dx^2} - \frac{(x^2+2x)}{x^2}\frac{dy}{dx} + \frac{(x+2)}{x^2}y = xe^x \longrightarrow (1)$$

This is of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

Here, $P = -\frac{(x^2+2x)}{x^2}$; $Q = \frac{(x+2)}{x^2}$; $R = xe^x$

Let $y = uv$ be the soln of (1),

To find u

$P + Qx = 0$

$= \frac{-(x^2+2x)}{x^2} + \frac{(x+2)x}{x^2}$

$= 0$

$y = x$ is soln of (2)

Then $y = x \cdot v \longrightarrow (3)$

diff. $\frac{dy}{dx} = x \cdot \frac{dv}{dx} + v \longrightarrow (4)$

$\frac{d^2y}{dx^2} = x \cdot \frac{d^2v}{dx^2} + \frac{dv}{dx} + \frac{dv}{dx}$

$= x \cdot \frac{d^2v}{dx^2} + 2\frac{dv}{dx} \longrightarrow (5)$

subs (3), (4), (5) in (1),

$$x\cdot\frac{d^2v}{dx^2}+2\cdot\frac{dv}{dx}-\frac{(x^2+2x)}{x^2}\left(x\frac{dv}{dx}+v\right)+\frac{(x+2)}{x^2}(x)=xe^x$$

$$x\cdot\frac{d^2v}{dx^2}+2\frac{dv}{dx}-\frac{x(x^2+2x)}{x^2}\frac{dv}{dx}-\frac{(x^2+2x)}{x^2}v+\frac{x+2}{x^2}(xu)=xe^x$$

$$x\cdot\frac{d^2v}{dx^2}+\frac{dv}{dx}-\left(\frac{x^2+2x}{x^2}+\frac{x^2}{x}\right)=xe^x.$$

$$x\cdot\frac{d^2v}{dx^2}+\frac{dv}{dx}(-x)=xe^x$$

$$\frac{d^2v}{dx^2}-\frac{dv}{dx}=e^x\longrightarrow (*)$$

Let $\frac{dv}{dx}=P$

$$\frac{d^2v}{dx^2}=\frac{dP}{dx}.$$

(*) becomes,

$$\frac{dP}{dx}-P=e^x$$

This is linear form.

$P_1=-1 \qquad Q_1=e^x$

$$P.e^{\int P_1dx}=\int Q_1e^{-\int P_1dx}$$

$$\boxed{e^{\int P_1dx}=e^{-\int dx}}\ \boxed{P_1=-1}$$

$$=e^{-x}$$

$$\boxed{e^{\int P_1dx}=e^{-x}}$$

$$Pe^{-x}=\int e^xe^{-x}dx+c_1$$

$$Pe^{-x}=\int e^{x-x}dx+c_1$$

$$\boxed{e^0=1}$$

$$Pe^{-x}=e^0x+c_1$$

$$Pe^{-x}=e^0x+c_1$$

$$P=(x+c_1)e^x$$

$$\frac{dv}{dx}=(x+c_1)e^x.$$

$$\frac{dv}{dx} = (x+c_1)e^x$$

$$\int dv = \int (x+c_1)e^x \cdot dx$$

$$v = \int x \cdot e^x + c_1 e^x \, dx + c_2$$

$$v = x \cdot e^x + e^x + c_1 e^x - x + c_2$$

$$v = xe^x + e^x(c_1 - 1) + c_2$$

$y = x^2 e^x + xe^x(c_1 - 1) + c_2 x$ is the Req. solution.

2. Solve, $4x^2 \frac{d^2y}{dx^2} + 4x^5 \frac{dy}{dx} + (x^8 + 6x^4 + 4)y = 0$

Sol'n:

G.T. $4x^2 \frac{d^2y}{dx^2} + 4x^5 \frac{dy}{dx} + (x^8 + 6x^4 + 4)y = 0$

$$\frac{d^2y}{dx^2} + x^3 \cdot \frac{dy}{dx} + \frac{(x^8 + 6x^4 + 4)}{4x^2} y = 0 \longrightarrow (1)$$

This is of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$.

$P = x^3$, $Q = \frac{1}{4}\left(x^6 - 6x^2 + \frac{4}{x^2}\right) y$ $\quad R = 0$

Let $y = uv$ be the soln of (1),

choose $u = e^{-\frac{1}{2}\int P dx}$

$$u = e^{-\frac{1}{2}\int x^3 dx}$$

$$= e^{-1/2 \frac{x^4}{4}}$$

$$= e^{-\frac{x^4}{8}}$$

(1) Reduces to,

$$\frac{d^2v}{dx^2} + Q_1 V = R_1 \longrightarrow (2)$$

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$ $\quad R_1 = \frac{R}{u}$

$$Q_1 = \frac{1}{4}\left(x^6 + 6x^2 + \frac{4}{x^2}\right) - \frac{1}{2}(3x^2) - \frac{x^9}{4}$$

$$= \frac{1}{4}\left(x^6 + 6x^2 + 4\right) - \frac{3x^2}{2} - \frac{x^4}{4}$$

$$= \frac{x^6}{4} + \frac{\cancel{6}^{3}x^2}{\cancel{4}_{2}} + \frac{1}{x^2} - \frac{3x^2}{2} - \frac{x^6}{4}$$

$$= \frac{x^6}{4} + \frac{1}{x^2} - \frac{x^6}{4}$$

$$Q_1 = \frac{1}{x^2}$$

$$\boxed{Q_1 = \frac{1}{x^2}}$$

(2) becomes,

$$\frac{d^2V}{dx^2} + \frac{1}{x^2}V = 0$$

we have,

$$V = CF + PI$$

<u>To find CF</u>

$$\left(D^2 + \frac{1}{x^2}\right)V = 0$$

<u>The AE is</u>

$$m^2 + \frac{1}{x^2} = 0$$

$$m^2 = \frac{-1}{x^2}$$

$$\boxed{m = \frac{1}{x}i}$$

$$c_1 \cos\frac{1}{x} + c_2 \sin\frac{1}{x}$$

$$PI = 0$$

$$V = c_1 \cos\frac{1}{x} + c_2 \sin\frac{1}{x} + 0$$

$$y = e^{-\frac{x^4}{4}}\left(c_1 \cos\frac{1}{x} + c_2 \sin\frac{1}{x}\right)$$

Ex : 1

XXIII.

$$x\frac{d^2y}{dx^2}+2(4x-1)\frac{dy}{dx}-(9x-2)y=x^3e^x.$$

soln :-

G.T.

$$x\cdot\frac{d^2y}{dx^2}+2(4x-1)\frac{dy}{dx}-(9x-2)y=x^3e^x.$$

$$\frac{d^2y}{dx^2}+\frac{2(4x-1)}{x}\frac{dy}{dx}-\frac{(9x-2)}{x}y=x^2e^x \longrightarrow (1)$$

$$0 \longrightarrow (2)$$

This is of the form, $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R.$

$P=\frac{2(4x-1)}{x}$ $Q=-\frac{9x-2}{x}$, $R=x^2e^x.$

Let $y=uv$ be the soln of (1)

To find u

$$1+P+Q=0$$

$$1+\frac{8x-2}{x}-\frac{9x+2}{x}=0$$

$$\frac{x+8x-2-9x+2}{x}=0$$

$$\frac{9x-2-9x+2}{x}=0 \longrightarrow 0$$

$\therefore y=e^x$ is soln of (2)

then, $y=e^x\cdot v \longrightarrow (3)$

$$\frac{dy}{dx}=e^x\cdot\frac{dv}{dx}+e^x\cdot v \longrightarrow (4)$$

$$\frac{d^2y}{dx^2}=e^x\cdot\frac{d^2v}{dx^2}+e^x\cdot\frac{dv}{dx}+e^x\cdot\frac{dv}{dx}+e^x v.$$

$$=e^x\cdot\frac{d^2v}{dx^2}+2e^x\frac{dv}{dx}+e^x\cdot v \longrightarrow (5)$$

sub ③④⑤ in ①.

$$e^x \cdot \frac{d^2v}{dx^2} + 2e^x \frac{dv}{dx} + e^x v + \frac{2(4x-1)}{x} \left(e^x \cdot \frac{dv}{dx} + e^x v \right) - \frac{9x-2}{x}\left(e^x - v\right) = x^3 e^x.$$

cancelling e^x throughout,

$$\frac{d^2v}{dx^2} + 2 \cdot \frac{dv}{dx} + v + \frac{8x-2}{2}\left(\frac{dv}{dx} + v\right) - \frac{9x-2}{x} v = x^3$$

$$\frac{d^2v}{dx^2} + 2 \cdot \frac{dv}{dx} + v + \frac{8x-2}{x} \frac{dv}{dx} + v\,\frac{8x-2}{x} - \frac{9x-2}{x} v = x^3$$

$$\frac{d^2v}{dx^2} + \frac{dv}{dx}\left(\frac{2x+8x-2}{2}\right) + v\left(\frac{8x-2-9x+12}{x}\right) = x^3$$

$$\frac{d^2v}{dx^2} + \left(\frac{10x-2}{x}\right)\frac{dv}{dx} = x^3$$

$$\frac{d^2v}{dx^2} + 2\left(\frac{10x-2}{x}\right)\frac{dv}{dx} = x^3 \longrightarrow (*)$$

let, $\frac{dv}{dx} = P$ $\qquad \frac{d^2v}{dx^2} = \frac{dP}{dx}$.

(*) becomes,

$$\frac{dP}{dx} + \frac{(10x-2)}{x} P = x^3.$$

this is liniar eqn.

$$P_1 = \frac{10x-2}{x} \qquad Q_1 = 3$$

$$P = e^{\int P_1 dx} = \int Q_1 e^{\int P_1 dx} dx + c.$$

$$e^{\int P_1 dx}$$

$$e^{\int P_1 dx} = e^{\int \left(10 - \frac{2}{x}\right) dx}$$

$$= e^{\left[10x - 2\log x\right)}$$

$$= e^{10x - \log x^2}$$

$$= e^{10x} \cdot e^{\log \frac{1}{x^2}}$$

$$IF = e^{10x}\frac{1}{x^2}$$

$$P\cdot\frac{e^{10x}}{x^2} = \int x^2\cdot\frac{e^{10x}}{x^2}\,dx + c$$

$$\frac{Pe^{10x}}{x^2} = \int e^{10x}dx + c$$

$$\frac{Pe^{10x}}{x^2} = \frac{e^{10x}}{10} + c_1$$

$$P = \frac{x^2\left[e^{10x}+c_1\right]}{e^{10x}(10)}$$

$$\frac{dv}{dx} = \frac{x^2 e^{10x}}{e^{10x}(10)} + \frac{x^2 c_1}{e^{10x}(10)}$$

$$\frac{dv}{dx} = \frac{x^2}{10} + \frac{x^2 c_1}{e^{10x}\cdot 10}$$

$$\int dv = \int \frac{x^2}{10} + \frac{x^2 c_1}{e^{10x}\,10}\,dx$$

$$v = \frac{x^3}{30} + c_1\int\frac{x^2}{e^{10x}}dx + c_2$$

Hence the soln,

$$y = e^x\left[\frac{x^3}{30} + c_2 + c_1\int\frac{x^2}{e^{10x}}dx\right]$$

2) $xy_2 + (1-x)y_1 - y = e^x$.

soln

G.T. $xy_2 + (1-x)y_1 - y = e^x$

$$y_2 + \frac{(1-x)}{x}y_1 - \frac{y}{x} = \frac{e^x}{x} \longrightarrow (1)$$

This is of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} - Qy = R.$$

$P = \frac{(1-x)}{x} \quad Q = \frac{1}{x} \quad R = \frac{e^x}{x}$.

Let $y = uv$ be the soln of ①

To find u:

$$1 + P + Q = 1 + \frac{1-x}{x} + \frac{1}{x}$$

$$= \frac{x+1-x-1}{x}$$

$$= 0$$

$y = e^x . v$ is the soln of ②

Then,

$$\frac{dy}{dx} = e^x . \frac{dv}{dx} + e^x . v \longrightarrow ④$$

$$\frac{d^2y}{dx^2} = e^x . \frac{d^2v}{dx^2} + e^x . \frac{dv}{dx} + e^x . \frac{dv}{dx} + e^x . v$$

$$= e^x . \frac{d^2v}{dx^2} + 2e^x . \frac{dv}{dx} + e^x . v \longrightarrow ⑤$$

subs. ③, ④, ⑤ in ①,

$$e^x . \frac{d^2v}{dx^2} + 2e^x . \frac{dv}{dx} + e^x . v + \frac{(1-x)}{x}\left(e^x . \frac{dv}{dx} + e^x v\right) - \frac{e^x v}{x} = \frac{e^x}{x}$$

cancelling e^x,

$$\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + v + \frac{(1-x)}{x}\left(\frac{dv}{dx} + v\right) - \frac{v}{x} = \frac{1}{x}$$

$$\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + v + \frac{(1-x)}{x}\frac{dv}{dx} + \frac{(1-x)}{x} v - \frac{v}{x} = \frac{1}{x}.$$

$$\frac{d^2v}{dx^2} + \frac{dv}{dx}\left(\frac{2x+1-x}{x}\right) + \left(\frac{x+1-x-1}{x}\right) v = \frac{1}{x}$$

$$\frac{d^2v}{dx^2} + \left(\frac{x+1}{x}\right)\frac{dv}{dx} = \frac{1}{x} \longrightarrow (*)$$

Let $\frac{dv}{dx} = P \qquad \frac{d^2v}{dx^2} = \frac{d^2P}{dx}$

(*) becomes,

$$\frac{d^2P}{dx} + \frac{(x+1)}{x} P = 1/x$$

this is liniar in form.

$$P_1 = \frac{(x+1)}{x} \quad Q = 1/x$$

$$e^{\int P_1 dx} = e^{\int \frac{x+1}{x} dx}$$

$$= e^{\int 1 + 1/x \, dx}$$

$$= e^{\int dx + \int \frac{dx}{x}}$$

$$= e^{x + \log x}$$

$$= e^x e^{\log x}$$

$$= x e^x$$

$$P x e^x = \int \frac{1}{x} x e^x dx + c_1$$

$$p x e^x = \int e^x dx + c_1$$

$$P x e^x = e^x + c_1$$

$$P = \frac{e^x}{x e^x} + c_1$$

$$P = \frac{1}{x} + c_1 / x e^x$$

$$\frac{dv}{dx} = \frac{1}{x} + \frac{c_1}{x e^x}$$

$$dv = \frac{1}{x} dx + \frac{c_1}{x e^x}$$

$$\int dv = \int \frac{1}{x} dx + \frac{c_1}{x e^x} + c_2$$

$$v = \log x + c_2 + c_1 \int \frac{e^{-x}}{x} dx$$

$$y = e^x \left[(\log x + c_2 + c_1 \int \frac{e^{-x}}{x} dx \right]$$

3, $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$ $\left(x+\frac{1}{x}\right)$ is a soln.

soln :-

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}=0$$

this is of the form $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$

$P=\frac{1}{x}$, $Q=-\frac{1}{x^2}$ $R=0$

$u=x+\frac{1}{x}$

$y=u\cdot v$

$\Rightarrow\left(x+\frac{1}{x}\right)v \longrightarrow ①$

$y=\left(x+\frac{1}{x}\right)v$

$$\frac{dy}{dx}=x\frac{dv}{dx}+v+\frac{1}{x}\frac{dv}{dx}+v\left(-\frac{1}{x^2}\right)$$

$$=x\cdot\frac{dv}{dx}+\frac{1}{x}\frac{dv}{dx}+\left(1-\frac{1}{x^2}\right)v \longrightarrow ②$$

$$\frac{dy}{dx}=\frac{dv}{dx}\left(\frac{x^2+1}{x}\right)+\left(\frac{x^2-1}{x^2}\right)v$$

$$\frac{d^2y}{dx^2}=\frac{dv}{dx}\left(\frac{x(2x)-(x^2+1)}{x^2}\right)+\left(\frac{x^2+1}{x}\right)\frac{d^2v}{dx^2}+\left(\frac{x^2-1}{x^2}\right)\frac{dv}{dx}$$
$$+v\left(\frac{x^2(2x)-(x^2-1)2x}{x^4}\right)$$

$$\frac{d^2y}{dx^2}=\frac{dv}{dx}\left(\frac{x^2-1}{x^2}\right)+\left(\frac{x^2+1}{x}\right)\frac{d^2v}{dx^2}+\frac{dv}{dx}\left(\frac{x^2-1}{x^4}\right)+v\left(\frac{2}{x^3}\right)$$

① $\Rightarrow$ $x^2\frac{d^2y}{dx^2}+\frac{x\,dy}{dx}-y=0$

$\div x^2$, $\frac{d^2y}{dx^2}+\frac{1}{2}\frac{dy}{dx}-\frac{y}{x^2}=0 \longrightarrow (*)$

①, ②, ③ in $(*)$

$$\left(\frac{x^2+1}{x}\right)\frac{d^2V}{dx^2}+\frac{dV}{dx}\left(\frac{x^2-1}{x^2}\right)+\frac{dV}{dx}\left(\frac{x^2-1}{x^2}\right)+\frac{2V}{x^3}+\frac{1}{x}\left(\frac{dV}{dx}\left(\frac{x^2+1}{2}\right)+\left(\frac{x^2-1}{x^2}\right)\right)$$

$$-\left(\frac{x^2+1}{x}\right)\frac{V}{x^2}=0$$

$$\left(\frac{x^2+1}{2}\right)\frac{d^2V}{dx^2}+\frac{dV}{dx}\left[\frac{x^2-1}{x^2}+\frac{x^2-1}{x^2}+\frac{x^2+1}{x^2}\right]+V\left[\frac{2}{x^3}+\frac{x^2-1}{x^3}-\frac{(x^2+1)}{x^3}\right]=0$$

$$\left(\frac{x^2+1}{2}\right)\frac{d^2V}{dx^2}+\frac{dV}{dx}\left[\frac{3x^2-1}{x^2}\right]=0$$

$$\left[\div\left(\frac{x}{x^2+1}\right)\right]\frac{d^2V}{dx^2}+\frac{dV}{dx}\left[\frac{3x^2-1}{x^2}\cdot\frac{x}{x^2+1}\right]=0$$

$$\frac{d^2V}{dx^2}+\frac{dV}{dx}\left[\frac{3x^2-1}{x^3+x}\right]=0 \longrightarrow (4)$$

Let $P=\frac{dV}{dx} \Rightarrow \frac{dP}{dx}=\frac{d^2V}{dx^2}$.

(4) becomes,

$$\frac{dP}{dx}+P\left(\frac{3x^2-1}{x^3+x}\right)=0$$

here

$$P=\frac{3x^2-1}{x^3+x}$$

$$e^{\int P\,dx}=e^{\int\frac{3x^2-1}{x^3+x}}$$

$$=e^{\frac{3}{2}\log(x^2+1)-\int\frac{1}{x(x^2+1)}}$$

$$=e^{\log(x^2+1)^{3/2}+\log\frac{\sqrt{x^2+1}}{x}}$$

$$=e^{\log\left(\frac{x^2+1}{x}\right)^2}$$

$$=\frac{(x^2+1)^2}{x}$$

$$P\left(\frac{x^2+1}{x}\right)^2=0+c_1$$

$$c_1=P\frac{(x^2+1)}{2}$$

$$\frac{dv}{dx} = \frac{x}{(x^2+1)^2}$$

$$dv = c_1 \int \frac{x}{(x^2+1)^2}\, dx$$

$$\int dv = c_1 \int \frac{x}{(x^2+1)^2}\, dx$$

$$v = c_1 \left(-\frac{1}{2(x^2+1)} \right) + c_2 \quad \left(\text{by } (ax+b)^n \right)$$

$$y = \frac{c_1}{x} + c_2 \left(\frac{x+1}{x} \right)$$

(27). $P_1 = -2,\ Q = 1$

we have $p e^{\int P_1 dx} = \int Q \cdot e^{\int P_1 dx} dx + c.$

$$p e^{-2x} = \int e^{-2x} dx + c$$

$$p \cdot e^{-2x} = \frac{e^{-2x}}{2} + c$$

$$p \cdot \frac{1}{e^{2x}} = \frac{e^{-2x}}{2} + c$$

$$= \frac{e^{-2x}}{2} e^{2x} + c \cdot e^{2x}$$

$$p = \frac{-1}{2} + c \cdot e^{2x}$$

$$\frac{dv}{dx} = \frac{-1}{2} + c \cdot e^{2x}$$

$$\int dv = \int \left(-\frac{1}{2} + c \cdot e^{2x}\right) dx$$

$$= \frac{-1}{2} x + c \cdot \frac{e^{2x}}{2} + c_1$$

$$v = \frac{-x + ce^{2x}}{2} + c_1$$

$$y = x\left(\frac{ce^{2x} - x}{2}\right) + 2c_1$$

$2y = x(ce^{2x} - x\ 4c_1)$ is the soln.

CHANGE OF INDEPENDENT VARIABLE :

Consider, $\frac{d^2y}{dx^2} + P \cdot \frac{dy}{dx} + Qy = R \longrightarrow (1)$

(1) $\Rightarrow$ Reduces to $\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$

where $P_1 = \dfrac{\frac{d^2z}{dx^2} + P \frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$ $\quad Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$ $\quad R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

1. solve $\frac{d^2y}{dx^2} + \frac{dy}{dx}(\tan x + y\cos^2 x = 0$

soln :-

$$\frac{d^2y}{dx^2} + \frac{dy}{dx}\tan x + y\cos^2 x = 0 \longrightarrow ①$$

this is of $\frac{d^2y}{dx^2} + \frac{dy}{dx}P + Qy = R$

here $P = \tan x \quad Q = \cos^2 x \quad R = 0.$

$$① \Rightarrow \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = 0 \longrightarrow ②$$

where, $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} \qquad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$

we choose z such that $P_1 = 0$, $Q_1 =$ a constant.

let $P_1 = 0$

$$\text{ie,} \quad \frac{d^2z}{dx^2} + \frac{\tan x \frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = 0$$

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$$\frac{d^2z}{dx^2} + \tan x\frac{dz}{dx} = 0 \longrightarrow ③$$

Put $u = \frac{dz}{dx}$

$$\frac{du}{dx} = \frac{d^2z}{dx^2}$$

$$③ \Rightarrow \frac{du}{dx} + (\tan x)u = 0$$

$$\frac{du}{dx} = -(\tan x)u$$

$$\frac{du}{u} = -\tan x\,dx$$

$$\frac{du}{u} + \tan x\,dx = 0$$

Integrating $\log u + \log\sec x = c$

$$\log u + \log\sec x = c$$

$$\log u = -\log\sec x$$

$$\log u = \log\frac{1}{\sec x}$$

$$u = \cos x$$

$$\frac{dz}{dx} = \cos x$$

$$\int dz = \int \cos x\,dx$$

$$z = \sin x.$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{\cos^2 x}{(\cos x)^2}$$

$$\boxed{Q_1 = 1}$$

$$② \Rightarrow \frac{d^2y}{dz^2} + y = 0$$

$$(D^2+1)y = 0$$

The soln is $y = CF$

<u>To find CF</u>

The AE is $m^2+1=0$

$$m^2 = -1$$

$$m = \pm i$$

Imaginary

$$CF = A\cos z + B\sin z$$

$$y = A\cos(\sin x) + B\sin(\sin x).$$

Ex : XXIII

B(i) $(1+x^2)y_2 + xy_1 + 2y = 0$ by changing independent <u>variable</u>.

<u>soln</u> G.T. $(1+x^2)y_2 + xy_1 + 2y = 0$

This is of $\frac{d^2y}{dx^2} + \frac{dy}{dx}P + Qy = R$.

$$y_2 + \frac{x}{1+x^2}y_1 + \frac{2}{1+x^2}y = 0 \longrightarrow ①$$

$$P = \frac{x}{1+x^2} \quad Q = \frac{2}{1+x^2} \quad R = 0$$

$$① \Rightarrow \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = 0 \longrightarrow ②$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$ $\quad Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$

we choose z such that $P_1 = 0$ (or) $Q_1 =$ a constant

let $P_1 = 0$

$$\frac{\dfrac{d^2z}{dx^2} + P.\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = 0$$

(ie) $\dfrac{d^2z}{dx^2} + \dfrac{x}{1+x^2}\dfrac{dz}{dx} = 0 \longrightarrow$ (3)

Put $u = \dfrac{dz}{dx}$

then $\dfrac{du}{dx} = \dfrac{d^2z}{dx^2}$

(3) $\Rightarrow$ $\dfrac{du}{dx} + \left(\dfrac{x}{1+x^2}\right)u = 0$

$$\frac{du}{dx} = -\left(\frac{x}{1+x^2}\right)u$$

$$\frac{du}{u} = -\frac{x}{1+x^2}dx$$

$$\frac{du}{u} + \frac{x}{1+x^2}dx = 0$$

Integrating, $\log u + \frac{1}{2}\log(1+x^2) = 0$

$$\log u = -\frac{1}{2}\log(1+x^2)$$

$$\log u = -\log(1+x^2)^{-1/2}$$

$$\log u = -\log\sqrt{1+x^2}$$

$$= \log\left(\sqrt{1-x^2}\right)^{-1}$$

$$= \frac{1}{\sqrt{1+x^2}} \Rightarrow \frac{dz}{dx}.$$

$$\int dz = \int \frac{1}{\sqrt{1+x^2}}\,dx$$

$$z = \sinh^{-1}x.$$

$$Q_1 = \frac{}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{\frac{2}{1+x^2}}{\left(\frac{1}{\sqrt{1+x^2}}\right)^2} = \frac{2}{\cancel{1+x^2}} \times \frac{\cancel{1+x^2}}{1}$$

$$Q = 2.$$

$$\frac{d^2y}{dz^2} + 2y = 0$$

$$(D^2+2)y = 0$$

the AE is $m^2+2=0$

$$\Rightarrow m^2 = -2$$

$$\boxed{m = \pm\sqrt{2}i}$$

$\therefore$ Imaginary.

$$y = c_1 \cos\sqrt{2}z + c_2 \sin\sqrt{2}z$$

$$y = c_1 \cos\sqrt{2}\left(\sinh^{-1}x\right) + c_2 \sin\sqrt{2}\left(\sinh^{-1}x\right)$$

METHOD OF VARIABLE OF PARAMETERS

Pg : 153

Ex 1

solve $\frac{d^2y}{dx^2} + n^2y = \sec nx.$

soln :-

G.T. $\frac{d^2y}{dx^2} + n^2y = \sec nx \longrightarrow (1)$

$$\frac{d^2y}{dx^2} + n^2y = 0 \longrightarrow (2)$$

the soln of (2) is $y = CF$

to find CF

$$(D^2+n^2)y = 0$$

the AE is $m^2+n^2=0$

$$m^2 = -n^2$$

$$\boxed{m = \pm in}$$

$\therefore$ Imaginary & conjugate Pairs

$y = A\cos nx + B\sin nx$

$A, B \rightarrow$ Constant

we assume, $y = A(x)\cos nx + B(x)\sin nx$ is soln of (1) $\longrightarrow$ (3)

where $A(x)$ & $B(x)$ fns of x

choose $A(x)$ & $B(x)$,

$$A_1 \cos nx + B_1 \sin nx = 0 \longrightarrow (4)$$

& $$A_1 \frac{d}{dx}(\cos nx) + B_1 \frac{d}{dx}(\sin nx) = \sec nx$$

$$-A_1 n \sin nx + B_1 n \cos nx = \sec nx \longrightarrow (5)$$

solve (4) & (5)

$$A_1 n \sin nx \cos nx + B_1 n \sin^2 nx = 0$$
$$-A_1 n \sin nx \cos nx + B_1 n \cos^2 nx = \sec nx \cos nx$$

$$B_1 n \sin^2 nx + B_1 n \cos^2 nx = \sec nx \cos nx$$

$$B_1 n (\sin^2 nx + \cos^2 nx) = \sec nx \cos nx$$

$$B_1 n = \sec nx \cos nx$$

$$n B_1 = \sec x \cos nx$$

$$B_1 = \frac{1}{n}$$

$$\frac{dB}{dx} = \frac{1}{n}$$

$$dB = \frac{1}{n} dx.$$

Integrating $B = \frac{x}{n} + c_1$

$$A_1 n \cos^2 nx + B_1 \sin nx \, n \cos nx = 0$$
$$-A_1 n \sin^2 nx + B_1 \sin nx \cos nx = \sec nx \cdot \sin nx$$
$$+ \qquad - \qquad -$$

$$nA_1 (\cos^2 nx + \sin^2 nx) = -\sec nx \sin nx.$$

$$nA_1 = -\sec nx \cdot \sin nx$$

$$nA_1 = -\frac{1}{\cos nx} \sin nx$$

$$nA_1 = -\tan nx$$

$$A_1 = -\frac{\tan nx}{n}$$

$$\frac{dA}{dx} = \frac{-1}{n} \tan nx$$

Integrating,

$$A = \frac{-1}{n}\int \tan nx \, dx$$

$$= \frac{-1}{n}\frac{\log \sec nx}{n} + c_2$$

$$= \frac{-1}{n^2}\log \sec nx + c_2$$

$$= \frac{1}{n^2}\log (\sec nx)^{-1} + c_2$$

$$= \frac{1}{n^2}\log \frac{1}{\sec nx} + c_2$$

$$= \frac{1}{n^2}\log \cos nx + c_2$$

$$A(x) = \frac{\log \cos nx}{n^2} + c_2$$

sub the value of $A(x)$ & $B(x)$ in ③,

$$y = \left(\frac{\log \cos nx}{n^2}\right)\cancel{\cos x}\ \frac{1}{x}\ \cos x + \frac{x}{n}\sin nx + c_1 + c_2$$

Eg solve by method of Variation of Parameter,

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x.$$

soln :-

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x \longrightarrow ①$$

$$\frac{d^2y}{dx^2} + 4y = 0 \longrightarrow ②$$

the soln of ② is $y = CF$

To find CF

$$(D^2+4)y = 0$$

The AE is

$$m^2+4 = 0$$

$$m^2 = -4$$

$$m = \pm 2i$$

$\therefore$ Imaginary

$CF = A\cos 2x + B\sin 2x.$ $\quad A, B \rightarrow$ Constants

We assume that,

$y = A(x)\cos 2x + B(x)\sin 2x$ is soln of ①, ——→ ③

choose $A(x), B(x)$ ∴

$$A_1 \cos 2x + B_1 \sin 2x = 0 \longrightarrow ④$$

$$A_1 \frac{d}{dx}(\cos 2x) + B_1 \frac{d}{dx}(\sin 2x) = 4\tan 2x.$$

$$-A_1 2\sin 2x + B_1 2\cos 2x = 4\tan 2x \longrightarrow ⑤$$

solve ④, ⑤

$$A_1 2\sin 2x\cos 2x + 2B_1 \sin^2 2x = 0$$

$$-A_1 2\sin 2x\cos 2x + 2B_1 \cos^2 2x = 4\tan 2x\cos 2x$$

$$2B_1(\sin^2 2x + \cos^2 2x) = 4\tan 2x\cos 2x$$

$$2B_1 = 4 \cdot \frac{\sin 2x}{\cos 2x} \times \cos 2x.$$

$$2B_1 = 4\sin 2x$$

$$B_1 = 2\sin 2x$$

$$\frac{dB}{dx} = 2\sin 2x.$$

Integrating,

$$\int dB = 2\int \sin 2x\,dx$$

$$B = \frac{-2\cos 2x}{2}$$

$$B = -\cos 2x + c_1$$

sub the value of B_1 in ②

$$A_1 \cos 2x + 2\sin^2 2x = 0$$

$$A_1 \cos 2x = -2\sin^2 2x$$

$$A_1 = \frac{-2\sin^2 2x}{\cos 2x}$$

$$A_1 = -2\sin 2x\tan 2x$$

$$\frac{dA}{dx} = -2\sin 2x\tan 2x$$

Integrate, $\int dA = -2\int \sin 2x\tan 2x\,dx$

$$A = -2\int \sin 2x \cdot \frac{\sin 2x}{\cos 2x}\,dx$$

$$= 2\int \frac{\sin 2x}{\cos 2x}$$

$$= \frac{-2}{\cos 2x} + 2\frac{\cos^2 2x}{\cos 2x}$$

$$\frac{dA}{dx} = \frac{-2}{\cos 2x} + 2\cos 2x$$

$$\frac{dA}{dx} = -2\sec 2x + 2\cos 2x$$

$$dA = -2\sec 2x + 2\cos 2x\, dx.$$

$$\int dA = \int -2\sec 2x\, dx + 2\int \cos 2x\, dx$$

$$A = -2\log(\sec 2x + \tan 2x)\cdot\frac{1}{2} + c_2 + 2\sin 2x\cdot\frac{1}{2}$$

$$A = -\log(\sec 2x + \tan 2x) + \sin 2x + c_2$$

$$y = -\log(\sec 2x + \tan 2x) + \sin 2x + c_2(\cos 2x)$$

$$+ \cos 2x + c_1(\sin 2x).$$

$$\frac{dA}{dx} = -2\sec 2x + 2\cos 2x$$

$dA = (-2\sec 2x + 2\cos 2x)\,dx$

$\int dA = \int -2\sec 2x\,dx + 2\int \cos 2x\,dx$

$A = -2\log(\sec 2x + \tan 2x)\cdot\frac{1}{2} + C_2 + 2\sin 2x\cdot\frac{1}{2}$

$= -\log(\sec 2x + \tan 2x) + \sin 2x + C_2$

$y = -\log(\sec 2x + \tan 2x) + \sin 2x + C_2(\cos 2x) +$

$- \cos 2x + C_1(\sin 2x)$ //.

EQUATIONS NOT CONTAINING y DIRECTLY :-

these eqns are of the form,

$$\frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right)$$

Put $p = \frac{dy}{dx}$ & $\frac{d^2y}{dx^2} = \frac{dy}{dx}$

PROBLM

EG : 1

solve $(1+x^2)\frac{d^2y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0$

soln :

G.T $(1+x^2)\frac{d^2y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0 \longrightarrow (1)$

Putting $\frac{dy}{dx} = P$ & $\frac{d^2y}{dx^2} = \frac{dP}{dx}$ in (1)

$(1+x^2)\frac{dP}{dx} + 1 + P^2 = 0$

$(1+x^2)\frac{dP}{dx} = -(1+P^2)$

$$\frac{dP}{1+P^2} = -\frac{dx}{(1+x^2)}$$

Integrating, $\int \frac{dP}{1+P^2} = -\int \frac{dx}{1+x^2}$

$\tan^{-1}P = -\tan^{-1}(x) + \tan^{-1}(c)$

$\tan^{-1}p + \tan^{-1}(x) = \tan^{-1}c$

Let

$A = \tan^{-1}p \qquad B = \tan^{-1}x$

$\tan A = p \qquad \tan B = x$

W.K.T $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(\tan^{-1}p + \tan^{-1}x) = \dfrac{p+x}{1-xp}$

$\tan^{-1}p + \tan^{-1}x = \tan^{-1}\left(\dfrac{p+x}{1-xp}\right)$

$\tan^{-1}\left(\dfrac{p+x}{1-xp}\right) = \tan^{-1}c$

$\dfrac{p+x}{1-xp} = c$

$p + x = c(1-xp)$

$p + x = c - cxp$

$p + cxp = c - x$

$p(1+cx) = c - x$

$p = \dfrac{c-x}{1+cx}$

$\dfrac{dy}{dx} = \dfrac{c-x}{1+cx}$

$dy = \dfrac{c-x}{1+x}\,dx$

Integrating,

$\int dy = \int\left(\dfrac{c-x}{1+cx}\right)dx.$

Consider,

$\dfrac{c-x}{1+cx}$

$$\begin{array}{r|l} & -\frac{1}{c} \\ cx+1 & -x + c \\ & -x - \frac{1}{c} \\ \hline & c + 1/c \end{array}$$

$\dfrac{c-x}{1+cx} = -\dfrac{1}{c} + \dfrac{c + 1/c}{1+cx}$

$$\frac{c-x}{1+cx} = -\frac{1}{c} + \frac{c^2+1}{c(1+cx)}$$

$$\int dy = \int\left(-\frac{1}{c} + \frac{c^2+1}{(1+cx)}\right) dx$$

$$y = -\frac{1}{c}\int dx + \frac{c^2+1}{c}\int \frac{1}{1+cx}\, dx$$

$$= -\frac{1}{c}x + \frac{c^2+1}{c}\,\frac{\log(1+cx)}{c} + c_1$$

$$y = -\frac{1}{c}x + \frac{c^2+1}{c^2}\log(1+cx) + c_1.$$

EQNS NOT CONTAINING "x" DIRECTLY

these eqns are of the form,

$$\frac{d^2y}{dx^2} = f\left(y, \frac{dy}{dx}\right)$$

Put $\boxed{\frac{dy}{dx} = P}$, $\boxed{\frac{d^2y}{dx^2} = P\cdot\frac{dP}{dy}}$

PRBLM

1. solve. $y(1-\log y)\frac{d^2y}{dx^2} + (1+\log y)\left(\frac{dy}{dx}\right)^2 = 0$

soln :-

G.T. $y(1-\log y)\frac{d^2y}{dx^2} + (1+\log y)\left(\frac{dy}{dx}\right)^2 = 0$

$$y(1-\log y)P\cdot\frac{dP}{dy} + (1+\log y)(P)^2 = 0$$

$$\frac{dP}{P} = -\frac{1+\log y}{y(1-\log y)}\, dy.$$

$\left[\begin{array}{l}\text{let,}\\ z = \log y\end{array}\right]$ $\quad \frac{dP}{P} = -\frac{1+z}{\cancel{y}(1-z)}\, dz$

$$\int\frac{dP}{P} = -\int\frac{1+z}{(1-z)}\, dz \longrightarrow (1)$$

let $z = \log y$

$y = e^z$

$dy = e^z dz$,

consider, $\frac{1+z}{1-z}$

$$\frac{1+z}{1-z} = -1 + \frac{z}{1-z}\,dz$$

$$1-z \overline{\big)\, 1+z}$$
$$\underline{1-z}$$
$$2z$$

$$= -\int \frac{1+z}{1-z} \Rightarrow \int -z\,dz - \int \frac{d}{1-z}\,dz.$$

$$= -z - 2\log(1-z)$$

$$-1 + \frac{2}{1-z}.$$

① $\Rightarrow$

$$\log p = z + 2\log(1-z) + \log c.$$
$$= z + \log(1-z)^2 + \log c.$$
$$\log p = z + \log c(1-z)^2$$
$$e^{\log p} = e^{z+\log c(1-z)^2}$$
$$p = e^z\, e^{\log^c (1-z)^2}$$

$z = \log y$

$e^z = y$

$$p = e^z . c(1-z)^2$$
$$p = cy(1-\log y)^2$$
$$\frac{dy}{dx} = cy(1-\log y)^2$$
$$\frac{dy}{y(1-\log y)^2} = c\,dx.$$
$$\int \frac{dy}{y(1-\log y)^2} = \int c\,dx.$$
$$\frac{1}{1-\log y} = cx + c.$$

Ex :- XXIII.

Pg. 157

B)

(ii) solve . $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - y\sin^2 x = 0$

soln:

G.T. $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - y \sin^2 x = 0 \longrightarrow$ ①

This is of the form, $\frac{d^2y}{dx^2} + P.\frac{dy}{dx} + Qy = R.$

$P = -\cot x, \quad Q = -\sin^2 x, \quad R = 0.$

① transforms to,

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = 0 \longrightarrow ②$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P \cdot \frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}.$$

$$\log p = z + \log c(1-z)^2$$

$$e^{\log p} = e^{z + \log(1-z)^2}$$

$$p = e^z \cdot e^{\log(1-z)^2}$$

$$p = e^z \cdot c(1-z)^2$$

$$p = cy(1-\log y)^2$$

$$\frac{dy}{dx} = cy(1-\log y)^2$$

$$\frac{dy}{y(1-\log y)^2} = c\,dx$$

$$\int \frac{dy}{y(1-\log y)^2} = \int c\,dx.$$

$$\frac{1}{1-\log y} = cx + c$$

Pg: 157

H.W

Ex:- XXIII.

B) (ii) solve: $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - y \sin^2 x = 0$

soln:-

S.T.

$$\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - y\sin^2 x = 0 \longrightarrow (1)$$

this is of the form,

$$\frac{d^2y}{dx^2} + P\cdot\frac{dy}{dx} + Qy = R.$$

$P = -\cot x, \quad Q = -\sin^2 x, \quad R = 0$

(1) transforms to,

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = 0 \longrightarrow (2)$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} \qquad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

we choose, z s.t. $P_1 = 0$, $Q_1 = $ a constant.

let $P_1 = 0$

$$\frac{\frac{d^2z}{dx^2} + \left(-\cot x\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2} = 0$$

$$\frac{d^2z}{dx^2} - \cot x\frac{dz}{dx} = 0 \longrightarrow (3)$$

Put $u = \frac{dz}{dx}$.

$$\frac{du}{dx} = \frac{d^2z}{dx^2}$$

(3) becomes,

$$\frac{du}{dx} - \cot x\, u = 0$$

$$\frac{du}{dx} = (\cot x)u$$

$$\frac{du}{u} = \cot x\, dx.$$

Integrating,

$$\int\frac{du}{u} = \int\cot x\, dx$$

$$\log u = \log\sin x$$

$\log u = \log \sin x.$

$$\boxed{u = \sin x.}$$

$$\int \frac{dz}{dz} = \int \sin x$$

$$\int dz = \int \sin x \, dx$$

$$\boxed{z = -\cos x}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{-\sin^2 x}{(\sin x)^2}$$

$$\boxed{Q_1 = -1}$$

(2) becomes,

$$\frac{d^2y}{dz^2} + y = 0$$

$$(D^2 - 1)y = 0$$

The sum is $y = CF$.

The AE is $(m^2 - 1) = 0$

$$m^2 - 1 = 0$$

$$m^2 = 1$$

$$\boxed{m = \pm 1}$$

The Roots are Real & Distinct

$$CF = c_1 e^{z} + c_2 e^{-z}$$

$$y = c_1 e^{z} + c_2 e^{-z}$$

$$y = c_1 e^{-\cos x} + c_2 e^{\cos x}.$$

Solve $4x^3\frac{d^2y}{dx^2}+6x^2\frac{dy}{dx}+\tilde{x}y=0$

Soln :-

G.T. $4x^3\cdot\frac{d^2y}{dx^2}+6x^2\frac{dy}{dx}+x^2y=0$

$$\frac{d^2y}{dx^2}+\frac{3}{2x}\frac{dy}{dx}+\frac{m^2}{4x^3}y=0 \longrightarrow (1)$$

$P=\frac{3}{2x}$, $Q=\frac{m^2}{4x^3}$, $R=0$.

(1) transforms,

$$\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+Q_1y=0 \longrightarrow (2)$$

$$P_1=\frac{\frac{d^2z}{dx^2}+P\cdot\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} , \quad Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

choose, $z=\int e^{-\int Pdx}$

$$=\int e^{-\int\frac{3}{2}xdx}$$

$$=e^{\log(x)^{\frac{-3}{2}}}dx.$$

$$z=\int x^{\frac{-3}{2}}dx$$

$$z=-2x^{3/2}$$

$$\frac{d^2z}{dx^2}=-2\left(\frac{-1}{2}\right)x^{-1/2-1}=x^{\frac{-3}{2}}$$

$$P_1=\frac{\frac{d^2z}{dx^2}+P\cdot\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$=\frac{\frac{-3}{x}x^{\frac{-5}{2}}+\frac{3}{2}x^{\frac{-3}{2}}}{x^{-3/2}}$$

$$=\frac{\frac{-3}{2}x^{-1}x^{-3/2}+\frac{3}{2}x\cdot x^{\frac{-3}{2}}}{x^{-3/2}}$$

$P_1 = 0$,

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{m^2}{4x^3}}{\left((x)^{-3/2}\right)^2}$$

$$\boxed{Q_1 = \frac{m^2}{4}}$$

(2) $\Rightarrow \frac{d^2y}{dz^2} + \frac{m^2}{4}y = 0$

Soln is $y = CF$

To find CF $= \left(D^2 + \frac{m^2}{4}\right)y = 0$

The AE is $\left(k^2 + \frac{m^2}{4}\right) = 0$

$$k^2 = -\frac{m^2}{4}$$

$$\boxed{k = \pm \frac{m}{2}}$$

$$CF = c_1 \cos\frac{m}{2}z + c_2 \sin\frac{m}{2}z$$

$$= c_1 \cos\frac{m}{2}\left(-2x^{-1/2}\right) + c_2 \sin\frac{m}{2}\left(\frac{2}{\sqrt{x}}\right)$$

$$= c_1 \cos\frac{-m}{\sqrt{x}} + c_2 \sin\frac{-m}{\sqrt{x}}.$$

$$y = c_1 \cos\frac{m}{\sqrt{x}} + c_2 \sin\frac{m}{\sqrt{x}}.$$

Eg : 2

solve $\frac{d^2y}{dx^2} - \frac{3x+1}{x^2-1}\frac{dy}{dx} + y\left\{\frac{6(x+1)}{(x-1)(3x+5)}\right\}^2 = 0$

soln :-

G.T. $\frac{d^2y}{dx^2} - \frac{3x+1}{x^2-1}\frac{dy}{dx} + y\left\{\frac{6(x+1)}{(x-1)(3x+5)}\right\}^2 = 0 \rightarrow (1)$

This is of the form,

$$\frac{d^2y}{dx^2} + P.\frac{dy}{dx} + Qy = R$$

$P = -\frac{3x+1}{x^2-1}$ $\quad Q = \left\{\frac{6(x+1)}{(x-1)(3x+5)}\right\}^2$ $\quad R = 0$

choose, $x = \int e^{-\int P dx} \longrightarrow (2)$

$$= \int e^{-\int -\frac{3x+1}{x^2-1} dx}$$

$$= \int e^{\int \frac{3x+1}{x^2-1} dx}$$

$$= -\int e^{\int \frac{3x+1}{+x^2-1} dx}$$

$$= e^{\int \frac{3x+1}{x^2-1} dx} \longrightarrow (3)$$

$$\frac{3x+1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$\frac{3x+1}{(x+1)(x-1)} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)}$$

$$3x+1 = A(x-1) + B(x+1)$$

Put $x = 1$,

$$B = 2$$

Put $x = -1 \longrightarrow -2 = -2A$

$$\boxed{A = 1}$$

(3) becomes,

$$\frac{3x+1}{x^2-1} = \frac{1}{x+1} + \frac{2}{x-1}$$

$$e^{\int \frac{3x+1}{x^2-1}} = e^{\int \frac{1}{x+1} + \frac{2}{x-1} dx + c}$$

$$= e^{\int \log(x+1) + 2\log(x-1) + \log c}$$

$$= e^{\int \log(x+1) + \log(x-1)^2 + \log c}$$

$$= e^{\int \log(x+1)(x-1)^2 + \log c}$$

$$e^{\int \frac{3x+1}{x^2-1} dx} = e^{\int \log(x+1)(x-1)^2 c}$$

$$= e^{\int (x+1)(x-1)^2}$$

② $\Rightarrow \int e^{\int (x+1)(x-1)^2} = \int (x+1)(x-1)^2\, du$

Let $u = x+1$ $\qquad \int dv = \int (x-1)^2$

$du = dx.$ $\qquad v = \frac{(x-1)^3}{3}.$

$$z = (x+1)\frac{(x-1)^3}{3} - \int \frac{(x-1)^3}{3}\, dx.$$

$$= \frac{(x+1)(x-1)^3}{3} - \frac{1}{3}\,\frac{(x-1)^4}{4}$$

$$= \frac{4(x+1)(x-1)^3 - (x-1)^4}{12}$$

$$= \frac{(x-1)^3}{12}\left[4(x+1) - (x-1)\right]$$

$$= \frac{(x-1)^3}{12}\left[4x + 4 - x - 1\right]$$

$$z = \frac{(x-1)^3(3x+5)}{12}$$

$$\frac{dz}{dx} = \frac{1}{12}\left[(x-1)^3(3) + (3x+5)\,3\,(x-1)^2\right]$$

$$= \frac{3(x-1)^2}{12}\left[(x-1) + 3x+5\right]$$

$$= \frac{(x-1)^2}{4}(4x+4)$$

$$= \frac{(x-1)^2\, 4(x+1)}{4}$$

$$\frac{dz}{dx} = (x-1)^2(x+1)$$

$$\frac{d^2z}{dx^2} = (x-1)^2(1) + (x+1)2(x-1)$$

$$= (x-1)\left[(x-1) + 2(x+1)\right]$$

$$= (x-1)\left[x - 1 + 2x + 2\right]$$

$$\frac{d^2z}{dx^2} = (x-1)(3x+1)$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P \cdot \frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{(x-1)(3x+1) + 1\left(\frac{-3x+1}{x^2-1}\right)\left((x-1)^2(x+1)\right)}{\left[(x-1)^2(x+1)\right]^2}$$

$$= \frac{(x-1)(3x+1) - \frac{3x+1}{(x+1)(x-1)}(x-1)^2(x+1)}{\left[(x-1)^2(x+1)\right]^2}$$

$$= \frac{(x-1)(3x+1) - 3x-1)\,(x-1)}{\left[(x-1)^2(x+1)\right]^2}$$

$$P_1 = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

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$$= \frac{\left[\frac{6(x+1)}{(x-1)(3x+5)}\right]}{\left[(x-1)^2(x+1)\right]^2}$$

$$= \frac{36(x+1)^2}{(x+1)^2(x-1)^4(x-1)^2(3x+5)^2}$$

$$Q_1 = \frac{36}{(x-1)^6(3x+5)^2}$$

(3) $\Rightarrow$

$$\frac{d^2y}{dz^2} + \frac{36}{(x-1)^6(3x+5)^2}\,y = 0$$

$$\frac{d^2y}{dz^2} + \frac{36y}{\left[(x-1)^3(3x+5)\right]^2} = 0 \qquad z = \frac{(x-1)^3(3x+5)}{12}$$

$$12z = (x-1)^3(3x+$$

$$\frac{d^2y}{dz^2} + \frac{36y}{(12z)^2} = 0$$

$$\frac{d^2y}{dx^2} + \frac{y}{4x^2} = 0$$

$$4x^2 \cdot \frac{d^2y}{dx^2} + y = 0$$

let $u = \log x$; $D = \frac{d}{du}$ & $x = e^u$

$$(4x^2D^2+1)y = 0$$

To find CF

$$(4x^2D^2+1)y = 0$$

$$4\left[\theta(\theta-1)+1\right]y = 0$$

$$(4\theta^2 - 4\theta + 1)y = 0$$

The AE is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)(2m-1) = 0$$

$$m = \tfrac{1}{2}, \tfrac{1}{2}.$$

$$CF = \left[c_1u + c_2\right]e^{\frac{1}{2}}$$

$$= \left[c_1 \log x + c_2\right]x^{\frac{1}{2}}$$

$$y = \left[c_1 \log \frac{(x-1)^3(3x+5)}{12} + c_2\right]\left[\frac{(x-1)^3(3x+5)}{12}\right]$$

H.W

1. $(D^2+1)y = \tan^2 x \longrightarrow (1)$

$(D^2+1)y = 0 \longrightarrow (2)$

soln of (2) is $y = CF$

To find CF

$$(D^2+1)y = 0$$

The AE is

$$m^2+1 = 0$$

$$m^2 = -1$$

$$\boxed{m = \pm i}$$

$y = A\cos x + B\sin x$, $A, B \rightarrow$ constant we assume that

$y = A(x)\cos x + B(x)\sin x \longrightarrow$ (3) is the sum of (1)

where, $A(x)$ & $B(x)$ such that

$A_1 \cos x + B_1 \sin x = 0 \longrightarrow$ (4)

$A_1 \frac{d}{dx}\cos x + B_1 \frac{d}{dx}\sin x = \tan x.$

$-A_1 \sin x + B_1 \cos x = \tan^2 x \longrightarrow$ (5)

solving (4) & (5),

$A_1 \cos x \sin x + B_1 \sin^2 x = 0$

$-A_1 \cos x \sin x + B_1 \cos^2 x = \tan^2 x \cdot \cos x$

$B_1 (\sin^2 x + \cos^2 x) = \tan^2 x \cos x$

$B_1 = \tan^2 x \cdot \cos x$

$\frac{dB}{dx} = \tan^2 x \cos x$

$dB = \tan^2 x \cdot \cos x \, dx$

$\int dB = \int \tan^2 x \cdot \cos x \cdot dx.$

$\int dB = \int \frac{\sin^2 x}{\cos^2 x} \cos x \cdot dx.$

$B = \int \frac{\sin^2 x}{\cos x} dx.$

$= \int \frac{1-\cos^2 x}{\cos x} dx.$

$= \int \frac{1}{\cos x} dx - \int \frac{\cos^2 x}{\cos x} dx$

$= \int \sec x \, dx - \int \cos x \, dx$

$B = \log(\sec x + \tan x) - \sin x + C_2$

The value B_1 subs in (4),

$A_1 \cos x + (\tan^2 x \cos x) \sin x = 0$

$A_1 \cos x + \frac{\sin^2 x}{\cos^2 x} \cdot \cos x \sin x = 0$

$$A_1 \cos x + \frac{\sin^3 x}{\cos x} = 0$$

$$\cos x \, A_1 = -\frac{\sin^3 x}{\cos x}$$

$$A_1 = \frac{-\sin^3 x}{\cos^2 x}$$

$$A_1 = -\tan^2 x \cdot \sin x$$

$$\frac{dA}{dx} = -\tan^2 x \cdot \sin x$$

$$dA = -\tan^2 x \, \sin x \cdot dx.$$

1. solve $yy'' = y'^2 - y'$

<u>soln</u>:

<u>G.T</u> $$y \cdot \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 - \left(\frac{dy}{dx}\right) \rightarrow (1)$$

$$y \cdot \frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{dy}{dx} - 1\right)$$

this eqn is not containing x directly

Let $\frac{dy}{dx} = P$ Then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \frac{d}{dy}\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \frac{dP}{dy} \cdot P.$$

$(1) \Rightarrow$ $$yP\frac{dP}{dy} = P^2 - P$$

$$y \cdot \frac{dP}{dy} = \frac{P^2 - P}{P}$$

$$y \cdot \frac{dP}{dy} = \frac{P(P-1)}{P}$$

$$y \cdot \frac{dP}{dy} = P - 1$$

$$\frac{dP}{P-1} = \frac{dy}{y}$$

$$= \frac{-dP}{1-P} = \frac{dy}{y}$$

$$\frac{dy}{y} + \frac{dP}{1-P} = 0$$

Integrating,

$$-\log(1-P) + \log y = \log c$$

$$\log(1-P)^{-1} + \log y = \log c$$

$$\log y(1-P)^{-1} = \log c$$

$$y(1-P)^{-1} = c$$

$$\frac{y}{1-P} = c$$

$$1-P = \frac{y}{c}$$

$$P = 1 - \frac{y}{c}$$

$$\frac{dy}{dx} = \frac{c-y}{c}$$

$$\frac{dy}{c-y} = \frac{dx}{c}$$

$$-\log(c-y) = \frac{1}{c}x + c_1$$

$$\frac{1}{c}x + \log(c-y) + c_1 = 0 \,//$$

TOTAL DIFFERENTIAL EQNS :-

Eqns of the type $\boxed{Pdx + Qdy + Rdz = 0}$ where P, Q, R are the functions of x, y, z are called total diff. eqns.

NECESSARY CONDITION FOR THE INTEGRABILITY :

$$\begin{vmatrix} P & Q & R \\ P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = 0$$

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$\Rightarrow$ which is the necessary condition for the integrability for the eqn.

$$Pdx + Qdy + Rdz = 0$$

RULE FOR SOLVING $\boxed{Pdx + Qdy + Rdz = 0}$

METHOD 1 :

Taking one variable constant

PROBLEM

1. verify the condition of integrability & solve $(y+z)\,dx + dy + dz = 0$

soln :

G.T. $(y+z)dx + dy + dz = 0 \longrightarrow (1)$

This is of the type $Pdx + Qdy + Rdz = 0$

$P = y+z \qquad Q = 1 \qquad R = 1$

$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial R}{\partial x} = 0$

$\frac{\partial P}{\partial z} = 1 \qquad \frac{\partial Q}{\partial z} = 0 \qquad \frac{\partial R}{\partial y} = 0$

we have the cdn of integrability is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$$\Rightarrow (y+z)(0-0) + 1(0-1) + 1(1-0) = 0$$

$$0 - 1 + 1 \Rightarrow 0$$

$\therefore$ cdn of integrability is satisfied.

we assume that, $x = a$ constant

then, $dx = 0$

the gn diff eqn $\Rightarrow dy + dz = 0$

Integrating, ~~dy + dz~~

$y + z =$ constant which is independent of y & z

$y + z = \phi(x)$

differentiating $dy + dz = du \longrightarrow$ ②

compare ① & ②

$(y+z)\,dx + d\phi = 0$

$\phi(x)\,dx + d\phi = 0$

$d\phi = -\phi\,dx$

$\frac{d\phi}{\phi} = -dx$

$\int \frac{d\phi}{\phi} = \int -dx$

$\log\phi + x = c_1$

$\log\phi = -x + c_1$

$\phi = e^{-x+c_1}$

$= e^{-x}\, e^{c_1}$

$\phi = ce^{-x}$ where $c = e^{c_1}$

$y + z = ce^{-x}$ is the soln of ①

2. solve $(y+z)dx + (z+x)dy + (x+y)dz = 0$

soln

$(y+z)dx + (z+x)dy + (x+y)dz = 0 \longrightarrow$ ①

this is of the type $Pdx + Qdy + Rdz = 0$.

$P = y+z$ $\quad$ $Q = z+x$ $\quad$ $R = (x+y)$

$\frac{\partial P}{\partial y} = 1$ $\quad$ $\frac{\partial Q}{\partial x} = 1$ $\quad$ $\frac{\partial R}{\partial x} = 1$

$\frac{\partial P}{\partial z} = 1$ $\quad$ $\frac{\partial Q}{\partial z} = 1$ $\quad$ $\frac{\partial R}{\partial y} = 1$

we have cdn of integrability is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$y + z\,(1-1) + z + x\,(1-1) + (x + y\,(1-1) = 0$

$= 0$

$\therefore$ Cnd of integrability is Verified.

we assume that $z = a$ constant

Then $dz = 0$

the gn diff eqn $\Rightarrow (y+z)dx + (z+x)dy = 0$

$$(y+z)\,dx = -(z+x)\,dy$$

$$\frac{dx}{z+x} = -\frac{dy}{y+z}$$

Integrating,

$$(y+z)\,dx = -(z+x)\,dy$$

$$\frac{dx}{z+x} = -\frac{dy}{y+z}$$

$$\frac{dx}{z+x} + \frac{dy}{y+z} = 0$$

Integrating,

$$\log(z+x) + \log(y+z) = \log c$$

$$\log(z+x)(y+z) = \log c$$

$$(z+x)(y+z) = c$$

$$(z+x)(y+z) = \phi(z)$$

Differentiating

$$(z+x)(dy+dz) + (y+z)(dx+dz) = \phi'(z)\,dz$$

$$(z+x)\,dy + (z+x)\,dz + (y+z)\,dx + (y+z)\,dz = \phi'\,dz$$

$$(y+z)\,dx + (z+x)\,dy + (z+x+y+z-\phi')\,dz = 0$$

$$(y+z)\,dx + (x+z)\,dy + (x+y+2z-\phi')\,dz = 0 \rightarrow (2)$$

comparing (1) & (2),

$$-\phi' + 2z = 0$$

$$\phi' = 2z$$

$$d\phi = 2z$$

Integrating, $\phi = z^2 + c$

$$(z+x)(y+z) = z^2 + c$$

$$yz + z^2 + xy + xz - z^2 = c$$

$xy + yz + xz = c$ is soln of (1)

1. solve $(y^2+yz)dx+(xz+z^2)dy+(y^2-xy)dz=0$

soln :

G.T $(y^2+yz)dx+(xz+z^2)dy+(y^2-xy)dz=0 \longrightarrow (1)$

this is of type $Pdx+Qdy+Rdz=0$

$P=y^2+yz \qquad Q=xz+z^2 \qquad R=y^2-xy$

$\frac{\partial P}{\partial y}=2y \qquad \frac{\partial Q}{\partial x}=z \qquad \frac{\partial R}{\partial x}=-y$

$\frac{\partial P}{\partial z}=y \qquad \frac{\partial Q}{\partial z}=x+2z \qquad \frac{\partial R}{\partial y}=2y-x$

we have the cdn of integrability,

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

$\Rightarrow y^2+yz(-2y)+xz+z^2(-y-y)+(y^2-xy)(2y+\cancel{x}-z)$

$\Rightarrow zy^2-2y^3+xy^2+yz^2-2y^2z+xyz-2yxz-2yz^2+2y^3$
$+zy^2-zy^2-xy^2y-xyz+2y^3-2y^2x$

$=0$

$\therefore$ cnd of integrability is verified.

we assume that $z=$ a constant,

then, $dz=0$.

The gn diff eqn $\Rightarrow (y^2+yz)dx+(xz+z^2)dy=$

$(y^2+yz)dx=-(xz+z^2)dy$

$$\frac{dx}{xz+z^2}=-\frac{dy}{y^2+yz}$$

$$\frac{dx}{z(x+z)}=-\frac{dy}{y(y+z)}$$

$$\frac{dx}{(x+z)}=\frac{-z\,dy}{y(y+z)}=0 \longrightarrow (2)$$

Integrating, $\log(x+z)$

consider

$$\frac{z}{y(y+z)}\,dy = \frac{A}{y} + \frac{B}{y+z}\,dy$$

$$z = A(y+z) + By$$

Put, $\boxed{y=-z}$

$$z = A(0) + -Bz$$

$$z = -Bz$$

$$\boxed{z=-1}$$

Put $\boxed{y=0}$

$$z = A(z) + 0$$

$$\boxed{A=1}$$

$$\frac{dy}{y} - \frac{dy}{y+z}$$

$$\log y - \log(y+z) = 0$$

$$\log y + \log(y+z)^{-1} = 0$$

$$\log y\,(y+z)^{-1} = 0$$

$$\log \frac{y}{y+z} = 0$$

(2) $\Rightarrow \log(x+z) + \log\dfrac{y}{y+z} = \log c$

$$\frac{(x+z)y}{y+z} = c - \phi(z) \longrightarrow (3)$$

Diff, (2)

$$\frac{(y+z)\left[(dx+dz)y + (x+z)dy\right] - y(x+z)(dy+dz)}{(y+z)^2} = \phi'(z)dz$$

$$(y^2+yz)\,dx + (xz+z^2)\,dy + dz\left(y^2 - xy - \phi'(z)\,(y+z)^2\right] = 0 \longrightarrow (4)$$

comparing (1) & (4)

$$\phi'(y+z)^2 dz = 0$$

$$(y+z)^2\,dz \neq 0$$

$$\phi'(z) = 0$$

Integrating, $\phi = c$ constant

$$\frac{(x+z)y}{y+z} = c$$

$(x+z)y = c(y+c)$ is the soln of (1)

METHOD 2 :

Rearranging the terms

PROBLE :-

Eg:

solve $(y+z)\,dx + dy + dz = 0$

soln

G.T $(y+z)\,dx + dy + dz = 0$

$P = y+z$ $\quad Q = 1$ $\quad R = 1$

$$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial R}{\partial x} = 0$$

$$\frac{\partial P}{\partial z} = 1 \qquad \frac{\partial Q}{\partial z} = 0 \qquad \frac{\partial R}{\partial y} = 0$$

we have cdn of integrability is,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$$(y+z)(0-0) + 1(0-1) + 1(1-0)$$

$$0 - 1 + 1 \Rightarrow 0$$

$\therefore$ verified

$$(y+z)\,dx = -dy - dz$$
$$dx = -\frac{(dy+dz)}{y+z}$$
$$dx + \frac{dy+dz}{y+z} = 0$$

Integrating,

$$\int dx + \int \frac{d(y+z)}{y+z} = 0$$
$$x + \log(y+z) = 0$$
$$\log(y+z) = -x + c$$
$$y+z = e^{-x} + c$$
$$y+z = e^{-x} e^{c}$$
$$y+z = e^{-x} c_1 \text{ is soln of } ①$$

<u>H.W</u>

1. $(y-z)\,dx + (z-x)\,dy + (x-y)\,dz = 0$

<u>soln</u>

<u>G.T</u>

$(y-z)\,dx + (z-x)\,dy + (x-y)\,dz = 0$

this is of the type $P\,dx + Q\,dy + R\,dz = 0$

$P = y - z \qquad Q = z - x \qquad R = x - y$

$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = -1 \qquad \frac{\partial R}{\partial x} = 1$

$\frac{\partial P}{\partial z} = -1 \qquad \frac{\partial Q}{\partial z} = 1 \qquad \frac{\partial R}{\partial y} = -1$

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial x}\right)$$
$$= (y-z)(1+1) + (z-1)(1+1) + (x-y)(1+1)$$
$$= 2y - z + \cancel{2z} - 2Q + 2y - 2y$$
$$= 0$$

$\therefore$ Verified

we assume that $z = a$ constant

then $dz = 0$

① $\Rightarrow (y-z)\,dx + (z-x)\,dy = 0.$

$$(y-z)dx = -(z-x)dy$$

$$\frac{dx}{z-x} = -\frac{dy}{y-z}$$

integrating,

$$\log(z-x) = -\log(y-z)\log c$$

$$\log \frac{y-x}{z-x} = \log c$$

$$\frac{y-x}{z-x} = \phi(z)$$

Differentiating,

$$\frac{(z-x)(dy-dz) - (y-x)(dz-dx)}{(z-x)^2} = \phi(z)\,dz$$

$$\frac{zdy - zdx + xdx - xdy - [y\,dz - ydx - zdx + zdx]}{(z-x)^2} = \phi'(z)dz$$

$$\left\{(y-z)dx + (z-x)dy + (x-y)dz = -\phi'(z) - x^2\right\} dz = 0$$

comparing ① & ②

$$\left[x-y-p'(z)(z-x)^2\right] = 0$$

$$-\phi'(z-x)^2 = 0$$

integrating,

$$\frac{y-z}{z-x} = c$$

$y - z = c(z-x)$ which is a Required soln

①

UNIT - IV

partial Differential equations

1) Form the partial differential equation by eliminating the arbitrary constants a and b from $z = (x+a)^2 + (y+b)^2 + c^2$

solution :

$$z = (x+a)^2 + (y+b)^2 + c^2 \rightarrow ①$$

Differential partially write to x and y we get

$$\frac{\partial z}{\partial x} = 2(x+a)\cdot(1) + 0 + 0$$

$$\frac{\partial z}{\partial x} = 2(x+a)$$

$$p = 2(x+a) \qquad \boxed{p/2 = x+a}$$

$$\frac{\partial z}{\partial y} = 2(y+b)(1) + 0 + 0$$

$$q = 2(y+b)$$

$$q = 2(y+b)$$

$$\boxed{q/2 = y+b}$$

Substituding the values of $x+a$ and $y+b$ in ①

$$z = (p/2)^2 + (q/2)^2 + c^2$$

$$z = \frac{p^2}{4} + \frac{q^2}{4} + c^2$$

$\boxed{4z = p^2 + q^2 + 4c^2}$ which is required partial differential equation.

2) Form the differential equation by eliminating the arbitrary constants a and b form $axy+b=z$

solution:

$$z=axy+b \rightarrow ①$$

Differential with write to x we get

$$\frac{\partial z}{\partial x}=ay$$

$$p=ay$$

Differential with write to y we get

$$\frac{\partial z}{\partial y}=ax$$

$$q=ax$$

Eliminating a Last two equation $p=y$, $q=x$
substituting p and q value ① equation we get.

$$\boxed{p/y=0} \quad \boxed{q/x=0}$$

$$z=q/x \cdot p/y$$

③ ~~Form~~

$$p/y=q/x$$

$$\boxed{px-qy=0}$$

which is the required partial differential equation

3. Form the partial differential equation by eliminating the arbitrary constants a and b from $z=axe^{y}+\frac{1}{2}a^{2}e^{2y}+b$

solution:

$$z=axe^{y}+\frac{1}{2}a^{2}e^{2y}+b \rightarrow ①$$

Differentiating partially write to x :

$$\frac{\partial z}{\partial x} = a(1)e^{y} \qquad \therefore \frac{\partial z}{\partial x} = p$$

$$p = ae^{y} \rightarrow ② \qquad a = pe^{-y}$$

Differentiating partially writo to y

$$\frac{\partial z}{\partial y} = axe^{y} + \tfrac{1}{\cancel{2}}a^{2}e^{2y}\cdot\cancel{(2)} \qquad \therefore \frac{\partial z}{\partial y} = q$$

$$q = axe^{y} + a^{2}e^{2y} \rightarrow ③$$

substituting the value of a in ③ we get

$$q = pe^{-y}xe^{y} + p^{2}e^{-2y}e^{2y}$$

$\boxed{q = px + p^{2}}$ which is the required partial differential equation.

4) Eliminate the arbitrary constant a, b, and c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and from a partial differential equation.

solution :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \rightarrow ①$$

Differentiating writo to x we get

$$\frac{2x}{a^2} + 0 + 2z/c^2 \cdot \frac{\partial z}{\partial x} = 0 \qquad \therefore \frac{\partial z}{\partial x} = p$$

$$x/a^2 + \frac{z}{c^2}p = 0 \rightarrow ②$$

Differentiating (1) partially writo to y we get

$$\frac{2y}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial y} = 0$$

$$\therefore \frac{\partial z}{\partial y} = q$$

$$\frac{y}{a^2} + \frac{z}{c^2} q = 0 \rightarrow ③$$

There are three constant and hence they cannot be eliminated from ① and ②, ③

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{x}{a^2} + \frac{z}{c^2} \cdot p = 0$$

Differentiating ② writo to we get

$$\frac{1}{a^2} + \frac{p^2}{c^2} + \frac{\partial^2 z}{\partial x^2} \cdot \frac{z}{c^2} = 0 \qquad \frac{\partial^2 z}{\partial x^2} = r$$

$$\frac{1}{a^2} + \frac{p^2}{c^2} + \frac{z}{c^2} \cdot r = 0 \rightarrow ④$$

Multiply ④ by x

$$\frac{x}{a^2} + \frac{p^2 x}{c^2} + \frac{zxr}{c^2} = 0 \rightarrow ⑤$$

Substituting it from ② we get

$$\frac{1}{c^2}[pz - xp^2 + xzr] = 0$$

$$\therefore pz - xp^2 + xzr = 0$$

$$pz = xp^2 + xzr$$

Differentiating ② w.r.t to y we get

$$\frac{1}{b^2} + \frac{q^2}{c^2} + \frac{z}{c^2} \cdot \frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{1}{b^2} + \frac{q^2}{c^2} + \frac{z}{c^2} \cdot s = 0 \rightarrow ⑥$$

multiply ⑥ by y we get

$$\frac{y}{b^2} + \frac{q^2 y}{c^2} + \frac{zy}{c^2} s = 0$$

substituting it from ③ we get

$$\frac{1}{c^2}[qz - yq^2 + yzt] = 0$$

$$qz - yq^2 + yzt = 0$$

$$\therefore qz = yq^2 + yzt$$

5. Eliminate the arbitrary function from $z = f(y/x)$ and from a partial differential equation.

solutions:

$$z = f(y/x) \rightarrow ①$$

Differentiating (1) partially write to x

$$\frac{\partial z}{\partial x} = f'(y/x) \cdot (-y/x^2)$$

$$p = -f'(y/x)(+y/x^2) \qquad -\frac{px^2}{y} = f(y/x)$$

Differentiating (1) partially write to y

$$\frac{\partial z}{\partial y} = f'(y/x) \cdot (1/x)$$

$$q = f'\left(\frac{y}{x^2}\right)(1/x)$$

$$qx = f'(y/x)$$

Finding $f'(y/x)$ from the first equation and substituting in the second we get

$$qx = \frac{-px^2}{y}$$

$$q = -\left(\frac{p\cancel{x^2}^{\,x}}{y}\right)(1/\cancel{x})$$

$$q = -\frac{px}{y}$$

$$qy = px$$

$px + qy = 0$ is the required partial differential equation.

6. from a partial differential equation by eliminating the arbitrary function f and g from $z = f(2x+y) + g(3x-y)$.

solution:

$$z = f(2x+y) + g(3x-y)$$

Differentiating partially w.r.t to x we get

$$\frac{\partial z}{\partial x} = f'(2x+y) \cdot 2(1) + g'(3x-y) \cdot 3(1)$$

$$\therefore \frac{\partial z}{\partial x} = p$$

$$p = 2f'(2x+y) + 3g'(3x-y) \rightarrow ①$$

Differentiating partially w.r.t to y we get

$$\frac{\partial z}{\partial y} = f'(2x+y) \cdot (1) + g'(3x-y)(-1)$$

$$q = f'(2x+y) - g'(3x-y) \rightarrow ② \quad \therefore \frac{\partial z}{\partial y} = q$$

Differentiating ① w.r.t to x

$$\frac{\partial^2 z}{\partial x^2} = 2[f''(2x+y) \cdot (2)] + 3[g''(3x-y) \cdot 3]$$

$$r = 4f''(2x+y) + 9g''(3x-y) \quad \therefore \frac{\partial^2 z}{\partial x^2} = r$$

Differentiating ② w.r.t to y

$$\frac{\partial^2 z}{\partial y^2} = f''(2x+y) \cdot (1) - g''(3x-y)(-1)$$

$$\therefore \frac{\partial^2 z}{\partial y^2} = s$$

$$s = f''(2x+y) + g''(3x-y)$$

solving for $f''(2x+y)$ and $g''(3x-y)$ from ③

and we get

$$f''(2x+y) = 1/5\,(9t-r)$$

$$g''(3x-y) = 1/5\,(r-4t)$$

Now differentiating ⓪ writ to x and y we get

$$\frac{\partial^2 z}{\partial x \partial y} = 2[f''(2x+y)\cdot(1)] + 3g''(3x-y)(-1)$$

$$s = 2f''(2x+y) - 3g''(3x-y) \rightarrow ⑤$$

substituting the values of $f''(2x+y)$ and $g''(3x-y)$ in ⑤ we get $f''(2x+y) = 1/5\,(9t-r)$; $g''(3x-y) = 1/5\,(r-4t)$

$$⑤ \Rightarrow s = 2 \times \frac{1}{5}(9t-r) - 3 \times 1/5\,(r-4t)$$

$$s = 2/5\,(9t-r) - 3/5\,(r-4t)$$

$$s = 1/5\,(18t-2r) - 3r + 12t$$

$$5s = 18t + 12t - 2r - 30r$$

$$5s = 30t - 5r$$

$$5s + 5r = 30t$$

$$5(r+s) = 30t \qquad \frac{30t}{5} = (r+s) \qquad \boxed{r+s = 6t}$$ which is required equation.

7) Eliminating the arbitrary function f and g form $z = f(x+ay) + g(x-ay)$ from a partial differential equation. (3)

Solution:

$$z = f(x+ay) + g(x-ay)$$

Differentiating partially write to x

$$\frac{\partial z}{\partial x} = f'(x+ay)(1) + g'(x-ay)(1)$$

$$p = f'(x+ay) + g'(x-ay) \rightarrow (1)$$

Differentiating partially y write to y we get

$$\frac{\partial z}{\partial y} = f'(x+ay)(a) + g'(x-ay)(-a)$$

$$q = af'(x+ay) - ag'(x-ay) \rightarrow (2)$$

Differentiating partially write to x (1) we get

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay)(1) + g''(x-ay)(1)$$

$$r = f''(x+ay) + g''(x-ay) \rightarrow (3)$$

Differentiating partially write to y (2) we get

$$\frac{\partial^2 z}{\partial y^2} = af''(x+ay)(a) - ag''(x-ay)(-a)$$

$$t = a^2 f''(x+ay) + a^2 g''(x-ay) \rightarrow (4)$$

$t = a^2[f''(x+ay)+g''(x-ay)]$

$t = a^2 r$

$t = a^2 r$ which is the required partial differential equation.

8) Form a partial differential equation by eliminating the arbitrary function c from $\phi(x+y+z; x^2+y^2-z^2) = 0$

$\phi(x+y+z, x^2+y^2-z^2) = 0$

Let $u = x+y+z$ $\quad \frac{\partial u}{\partial x} = 1 + \frac{\partial z}{\partial x} \quad \frac{\partial u}{\partial y} = 1 + \frac{\partial z}{\partial y}$

$v = x^2+y^2-z^2$ $\quad \partial v/\partial y = 2y - 2z\,\partial z/\partial y$

$\therefore$ The given equation becomes $\quad \frac{\partial v}{\partial x} = 2x - 2z\frac{\partial z}{\partial x}$

$\phi(u, v) = 0 \rightarrow (1)$

Differentiating (1) partially write to x any y we get

formula:

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right] = 0$$

$$\frac{\partial \phi}{\partial u}[1+p] + \frac{\partial \phi}{\partial v}[2x - 2zp] = 0 \rightarrow (2)$$

similarly differentiating (1) partially write to y we get

$$\frac{\partial \phi}{\partial u}[1+q] + \frac{\partial \phi}{\partial u}[2y - 2zq] \rightarrow (3)$$

$$② \Rightarrow \frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}} = \frac{-2(x-zp)}{1+p}$$

$$\left(\frac{\partial \phi / \partial u}{\partial \phi / \partial v}\right) = \frac{-2(x-zp)}{1+p} \rightarrow ④$$

$$③ \Rightarrow \left(\frac{\partial \phi / \partial u}{\partial \phi / \partial v}\right) = \frac{-2(y-zq)}{1+q} \rightarrow ⑤$$

From ④ and ⑤ we get:

$$-\cancel{2}\left(\frac{x-zp}{1+p}\right) = -\cancel{2}\left(\frac{y-zq}{1+q}\right)$$

$$(x-zp)(1+q) = (y-zq)(1+p)$$

$$x + xq - zp - zpq = y + py - zq - zpq$$

$$y + py - zq - zpq - x - xq + zp + zpq = 0$$

$(y+z)p - (x+z)q = x-y$ is the required partial differential equation.

Lagrange's theorem

1) solve: $2p - 3q = 1$

solution:

$$2p - 3q = 1$$

This is of the form $Pp + Qq = R$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{2} = \frac{dy}{-3} = \frac{dz}{1}$$

$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1}$$

king

$$\frac{dx}{2} = \frac{dy}{3}$$

$$3dx = 2dy$$

$\int$ing

$$3\int dx = 2\int dy$$

$3x - 2y = a$ is one solution.

auxillary taking

$$\frac{dy}{3} = \frac{dz}{1}$$

$$dy = 3dz$$

$\int$ing

$$\int dy = 3\int dz$$

$$y = 3z + b$$

$y - 3z = b$ is another solution.

$\therefore$ The general solution is $\phi(3x-2y, y-3z$

2) Find the general solution of $zp + x = 0$

$$zp + x = 0$$

The given differential equation can be writte in the from $P_p + Q_q = R$

$P = z$, $Q = 0$ and $R = -x$ (1)

$\therefore$ The auxillary equation is

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$$

Taking $\frac{dx}{z} = \frac{dz}{-x}$ we get

$\int$ing $\quad x\,dx + z\,dz = 0$

$$x\int dx + z\int dz = 0$$

$$\frac{x^2}{2} + \frac{z^2}{2} = c$$

$x^2 + z^2 = a$ where $a = 2c$ is one solution.

taking $\frac{dy}{0} = 0$

$$dy = 0$$

$\int$ing

$$\int dy = 0$$

$$y = b$$

$\therefore \phi(x^2+z^2, y)$ is the general solution.

3) Solve $x^2p + y^2q = z^2$

This is of the form $Pp + Qq = R$

The auxillary equation is $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$

Taking $\frac{dx}{x^2} = \frac{dy}{y^2}$ we get

$\int$ing

$$\int 1/x^2\, dx = \int 1/y^2\, dy$$

$x^{-1} = y^{-1} + a$

$-1/x = -1/y + a$

Taking

$\frac{dy}{y^2} = \frac{dz}{z^2}$ we get

$\int$ing $\int 1/y^2 dy = \int 1/z^2 dz$

$y^{-1} = z^{-1} + b$

$-1/y = -1/z + b$

$\frac{1}{y} - \frac{1}{z} + b = 0$

$\therefore \phi\left[+1/x - 1/y \,,\, 1/y - 1/z\right] = 0$ is general solution

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4) Solve $p\cot x + q\cot y = \cot z$

This is of the form $P_p + Q_q = R$

The auxillary equation is

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

Taking

$\frac{dx}{\cot x} = \frac{dy}{\cot y}$ we get

$\tan x\,dx = \tan y\,dy$

$\int$ing

$\int \tan x\,dx = \int \tan y\,dy$

$\log \sec x = \log \sec y + \log a$

$\log \sec x - \log \sec y = \log a$

$\log\left(\frac{\sec x}{\sec y}\right) = \log a$

$\frac{\sec x}{\sec y} = a$

$\frac{\frac{1}{\cos x}}{1/\cos y} = a$ $\quad \frac{1}{\cos x} \times \frac{\cos y}{1} = a$

$a = \frac{\cos y}{\cos x}$ is one solution.

similarly taking

$\frac{dy}{\cot y} = \frac{dz}{\cot z}$

$\tan y\, dy = \tan z\, dz$ we get

$\int$ing $\int \tan y\, dy = \int \tan z\, dz$

$\log \sec y = \log \sec z + \log b$

$\log \sec y - \log \sec z = \log b$

$\log\left(\frac{\sec y}{\sec z}\right) = \log b$

$\frac{\sec y}{\sec z} = b$

$\frac{\frac{1}{\cos y}}{1/\cos z} = b$

$$\frac{1}{\cos y} \times \frac{\cos z}{1} = b$$

$$\frac{\cos z}{\cos y} = b$$

$\therefore$ The general solution $\phi\left[\frac{\cos y}{\cos x}, \frac{\cos z}{\cos y}\right] = 0$

---

5) Solve : $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

solution :

$$p\sqrt{x} + q\sqrt{y} = \sqrt{z}$$

This is of the form $Pp + Qq = R$

The auxillary equation

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Taking $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$ we get

$\int$ing

$$\int \frac{1}{\sqrt{x}} dx = \int \frac{1}{\sqrt{y}} dy$$

$$2\sqrt{x} = 2\sqrt{y} + a$$

$\sqrt{x} - \sqrt{y} = a$ is one solution.

Taking

$\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$ we get

$\int$ing

$$\int \frac{1}{\sqrt{y}} dy = \int \frac{1}{\sqrt{z}} dz$$

$$2\sqrt{y} = 2\sqrt{z} + b$$

$\sqrt{y} - \sqrt{z} = b$ is another solution

$\therefore$ The general solution is $\phi(\sqrt{x}-\sqrt{y}, \sqrt{y}-\sqrt{z}) = 0$

6) Solve $y^2 z p + x^2 z p = xy^2$

This one of from $P_p + Q_q = R$

The auxillary equation is $\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$

Taking $\frac{dx}{y^2 z} = \frac{dy}{x^2 z}$

$$x^2 dx = y^2 dy$$

$\int$ing

$$\int x^2 dx = \int y^2 dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + a$$

$$\frac{1}{3}(x^3 - y^3) = a$$

$x^3 - y^3 = a$ is one solution.

Taking

$$\frac{dx}{y^2 z} = \frac{dz}{xy^2}$$

$$x\,dx = z\,dz$$

$\int$ing

$$\int x\,dx = \int z\,dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + b$$

$\therefore x^2 - z^2 = b$ is another solution.

The general solution is $\phi(x^3 y^3, x^2 - z^2) = 0$

7) Find the general solution of $p + 3q = 5z + \tan(y-3x)$

solution:

$$p + 3q = 5z + \tan(y-3x) \rightarrow ①$$

This is of the form $Pp + Qq = R$

The auxillary equation $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$

Taking $\frac{dx}{1} = \frac{dy}{3}$ we get

$$3dx = dy$$

$\int$ing

$$3\int dx = \int dy$$

$$-a = y - 3x \quad a - 3x = -y$$

$$3x = y + a \qquad y - 3x = a \rightarrow ②$$

$-3x + y = a$ is one solution.

Taking

$$\frac{dx}{1} = \frac{dz}{5z + \tan(y-3x)} \quad \text{from (2) } a = y - 3x$$

$$\frac{dx}{1} = \frac{dz}{5z + \tan a}$$

$\int$ing

$$\int dx = \int \frac{dz}{5z + \tan a}$$

$$\int dx = \frac{1}{5}\int \frac{5dz}{5z + \tan a}$$

$$x = \frac{1}{5}\log(5z + \tan a) + b$$

$5x - \log(5z + \tan a) = b$ is another

solution.

$\therefore$ the general solution is $\phi(y-3x, 5x - \log(5z+\tan a)) = 0$

8) solve $(y^2+z^2)p - xyq - xz = 0$

solution :

$$(y^2+z^2)p - xyq - xz = 0$$

This is of the form $Pp + Qq = R$

The auxillary equation

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$$

Taking $\frac{dy}{-xy} = \frac{dz}{-xz}$

$$\frac{dy}{y} = \frac{dz}{z}$$

Sing we get

$$\int \frac{1}{y} dy = \int \frac{1}{z} dz$$

$$\log y = \log z + \log a$$

$$\log y - \log z = \log a$$

$$\log\left[\frac{y}{z}\right] = \log a$$

$$\therefore \frac{y}{z} = a$$

Taking x, y, z as Lagrangian multiplies we get

$$\frac{x\,dx}{y^2+z^2} = \frac{y\,dy}{-xy^2} = \frac{z\,dz}{-xz^2} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

$$x\,dx+y\,dy+z\,dz=0$$

$\int$ing we get

$$\int x\,dx+\int y\,dy+\int z\,dz=0$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=b$$

$x^2+y^2+z^2=b$ is another solution.

$\therefore$ The general solution is $\phi(y/z, x^2+y^2+z^2)=0$.

---

9. Find the general solution of

$$x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$$

solution:

this is of the form $Pp+Qq=R$

the auxillary equation

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

Taking the Lagrangian multiplies as x, y, z we get each ratio (1) to equal

$$\frac{x\,dx}{x^2(y^2-z^2)} = \frac{y\,dy}{y^2(z^2-x^2)} = \frac{z\,dz}{z^2(x^2-y^2)}$$

$$\frac{x\,dx+y\,dy+z\,dz}{\cancel{x^2y^2}-\cancel{x^2z^2}+\cancel{y^2z^2}-\cancel{x^2y^2}+\cancel{z^2x^2}-\cancel{z^2y^2}}=0$$

$x\,dx + y\,dy + z\,dz = 0$ ⓑ

Sing we get

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a$$

$x^2+y^2+z^2 = a$ is one solution.

Taking the Lagrangian multiplies $1/x, 1/y, 1/z$ we get

$$= \frac{dx/x + \frac{dy}{y} + \frac{dz}{z}}{\frac{\cancel{x}(y^2-z^2)}{\cancel{x}} + \frac{\cancel{y}(z^2-x^2)}{\cancel{y}} + \frac{\cancel{z}(x^2-y^2)}{\cancel{z}}} = 0$$

$$= \frac{dx/x + dy/y + dz/z}{y^2-z^2+z^2-x^2+x^2-y^2} = 0$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Sing

$$\int 1/x\,dx + \int 1/y\,dy + \int 1/z\,dz = 0$$

$$\log x + \log y + \log z = \log b$$

$$\log(xyz) = \log b$$

$xyz = b$ is another solution.

$\therefore$ The general solution is $\phi(x^2+y^2+z^2, xyz) = 0$

10. solve : $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

solution.

$$x^2(y-z)p + y^2(z-x)q = z^2(x-y)$$

This is of the form $P_p + Q_q = R$

The auxillary equation

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \rightarrow (1)$$

Taking the Lagragian multipliers $1/x, 1/y, 1/z$

we get each ratio of (1) equal to

$$\frac{dx/x + \frac{dy}{y} + \frac{dz}{z}}{\frac{x^2(y-z)}{x} + \frac{y^2(z-x)}{y} + \frac{z^2(x-y)}{z}} = 0$$

$$\frac{dx/x + dy/y + dz/z}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{xy} + \cancel{zx} - \cancel{zy}} = 0$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$\int$ ing we get

$$\int \frac{1}{x} dx + \int \frac{1}{y} dy + \int \frac{1}{z} dz = 0$$

$$\log x + \log y + \log z = \log a$$

$$\log(xyz) = \log a$$

$xyz = a$ is one solution.

Taking the Lagragian multipliers as $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$

we get each ratio of ①

$$\frac{dx/x^2 + dy/y^2 + dz/z^2}{\frac{x^2(y-z)}{x^2} + \frac{y^2(z-x)}{y^2} + \frac{z^2(x-y)}{z^2}} = 0$$

$$\frac{dx/x^2 + dy/y^2 + dz/z^2}{y-z+z-x+x-y} = 0$$

$$\therefore dx/x^2 + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

Sing we get

$$\int 1/x^2 dx + \int 1/y^2 dy + \int 1/z^2 dz = 0$$

$$\int x^{-2} dx + \int y^{-2} dy + \int z^{-2} dz = 0$$

$$\frac{x^{-1}}{-1} + \frac{y^{-1}}{-1} + \frac{z^{-1}}{1} = b$$

$$-1/x - 1/y - 1/z = b$$

$\frac{1}{x} + 1/y + 1/z = b$ is another solution.

$\therefore$ The general solution is $\phi\left[xyz \cdot 1/x + 1/y + 1/z\right] = 0$

11) solve: $\left[\frac{y-z}{yz}\right]p + \left[\frac{z-x}{zx}\right]q = \frac{x-y}{xy}$

solution:

This is of the form $P_p + Q_q = R$

Multiply by xyz $\quad x(y-z)p + y(z-x)q = z(x-y)$

The auxillary equation

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Taking the Lagragian multipliers as 1, 1, 1 we get

$$\frac{dx+dy+dz}{xy-xz+yz-yx+zx-zy} = 0$$

$$dx+dy+dz=0$$

$\int$ing we get

$$\int dx+\int dy+\int dz=0$$

$x+y+z=a$ is one solution.

Also taking the Lagragian multipliers as $1/x, 1/y, 1/z$ we get each ratio in (1)

$$\frac{dx/x + dy/y + dz/z}{\frac{x(y-z)}{x} + \frac{y(z-x)}{x} + \frac{z(x-y)}{x}} = 0$$

$$\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y-z+z-x+x-y} = 0$$

$$dx/x + dy/y + dz/z = 0$$

$\int$ing we get

$$\int 1/x\,dx + \int 1/y\,dy + \int 1/z\,dz = 0$$

$$\log x + \log y + \log z = \log b$$

$$\log(xyz) = \log b$$

$$xyz = b$$

$\therefore$ The general solution is $\phi[x+y+z, xyz]=0$

12. solve: $(x^2-y^2-z^2)p+2xyq=2xz$

solutions:

This is of the form $Pp+Qq=R$

The auxillary equation is

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$$

Taking $\frac{dy}{2xy}=\frac{dz}{2xz}$

$$\frac{dy}{y}=\frac{dz}{z}$$

$\int$ ing weget

$$\int 1/y\, dy=\int 1/z\, dz$$

$$\log y=\log z+\log a$$

$$\log y-\log z=\log a$$

$$\log\left(\frac{y}{z}\right)=\log a$$

$\frac{y}{z}=a$ is one solution.

Taking Lagragin multipliers as x, y, z weget

$$\frac{xdx+ydz+zdz}{(x^3-xy^2-xz^2)+2xy^2+2xz^2}=0$$

$$\frac{xdx+ydz+zdz}{x^3+xy^2+xz^2}=0$$

$$\frac{xdx+ydz+zdz}{x(x^2+y^2+z^2)}=0$$

Taking $\frac{dy}{2xy} = \frac{xdx + ydy + zdz}{x(x^2+y^2+z^2)}$

$$\frac{dy}{2y} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

$$\frac{dy}{y} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

$\int$ ing we get

$$\int 1/y \, dy = \int \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

$$\log y = \log(x^2+y^2+z^2) + \log b$$

$$\log y - \log(x^2+y^2+z^2) = \log b$$

$$\log\left[\frac{y}{x^2+y^2+z^2}\right] = \log b$$

$\frac{y}{x^2+y^2+z^2} = b$ is another solution.

$\therefore$ The general solution is $\phi\left[y/z, \frac{y}{x^2+y^2+z^2}\right] = 0$

13) solve: $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$

solution:

This is of the form $Pp + Qq = R$

The auxillary equation is

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \rightarrow (1)$$

Taking Lagragian multipliers $1/x, 1/y, 1/z$ we get

each ratio (1) we get.

$$\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{\frac{x(y^2+z)}{x}-\frac{y(x^2+z)}{y}+\frac{(x^2-y^2)z}{z}}=0$$

$$\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2+z-x^2-z+x^2-y^2}=0$$

$$\therefore \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

$\int$ing we get

$$\int \tfrac{1}{x}dx+\int \tfrac{1}{y}dy+\int \tfrac{1}{z}dz=0$$

$$\log x+\log y+\log z=\log a$$

$$\log(xyz)=\log a$$

$xyz=a$ is one Solution.

Taking the Lagragian multipliers as $x, y, -1$ we get

$$\frac{xdx+ydy-zdz}{x^2(y^2+z)-y^2(x^2+z)-z(x^2-y^2)}=0$$

$$\frac{xdx+ydy+zdz}{x^2y^2+x^2z-x^2y^2-y^2z-zx^2+zy^2}=0$$

$$\therefore xdx+ydy+zdz=0$$

$\int$ing we get

$$\int xdx+\int ydy-\int zdz=0$$

$$\frac{x^2}{2}+\frac{y^2}{2}-z=c$$

$x^2+y^2-2z=b$ is another solution

$\therefore$ The general solution is $\phi(xyz, x^2+y^2-z)=0$

4) solve: $(x+y)zp+(x-y)zq=x^2+y^2$

solution:

this is of the form $P_p+Q_q=R$

The auxillary equation

$$\frac{dx}{(x+y)z}=\frac{dy}{(x-y)z}=\frac{dz}{x^2+y^2} \rightarrow ①$$

Taking the Lagragian multipliers as x, y, z wege each ratio ① we get:

$$\frac{xdx+ydy+zdz}{x(x+y)z+y(x-y)z+z(x^2+y^2)}=0$$

$$\frac{xdx+ydy+zdz}{x^2z+xyz+xyz-y^2z+zx^2+zy^2}=0$$

$$\frac{xdx+ydy+zdz}{2x^2z+2xyz}=0$$

$$\frac{xdx+ydy+zdz}{2xz(x+y)}=0$$

Taking $\frac{dx}{(x+y)z}=\frac{xdx+ydy+zdz}{2x\cancel{z}(\cancel{x+y})}$

$2xdx=xdx+ydy+zdz$

$\therefore\ 2xdx - xdx - ydy - 2dz = 0$ ⑧

$\therefore\ xdx - ydy - zdz = 0$

$\int$ing we get

$$\int xdx - \int ydy - \int zdz = 0$$

$$\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = a$$

$$x^2 - y^2 - z^2 = a$$

$x^2 - y^2 - z^2 = a$ is one solution.

Taking Lagragian multipliers $y, x, -z$ we get each ratio of (1)

$$\frac{ydx + xdy - zdz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = 0$$

$$\frac{ydx + xdy - zdz}{yzx + y^2z + x^2z - xyz - zx^2 - zy^2} = 0$$

$$ydx + xdy - zdz = 0$$

$$ydx + xdy = zdz$$

$$zdz = d(xy)$$

$\int$ing we get

$$\int zdz = \int d(xy)$$

$$\frac{z^2}{2} = xy + b$$

$$z^2/2 - xy = b$$

$z^2-2xy=b$ is another solution

The general solution is $\phi[x^2-y^2-z^2,\ z^2-2xy]=0$

15 solve : $(y+z)p+(z+x)q=x+y$

solution:

This is of the form $Pp+Qq=R$

The auxillary equation

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}$$

Each ratio can be written as

$$\frac{dx+dy+dz}{y+z+z+x+x+y}$$

$$\frac{dx+dy+dz}{2y+2x+2z}=\frac{dx+dy+dz}{2(x+y+z)}$$

Taking

$$\frac{dx-dy}{y+z-\cancel{z}-x}=\frac{dy-dz}{z+\cancel{x}-x-y}$$

$$\frac{dx-dy}{y-x}=\frac{dy-dz}{z-y}$$ we have

$$\frac{d(x-y)}{\cancel{-}(x-y)}=\frac{d(y-z)}{\cancel{-}(y-z)}$$

$$\frac{d(x-y)}{x-y}=\frac{d(y-z)}{y-z}$$

Jing we get

$$\int \frac{d(x-y)}{x-y} = \int \frac{d(y-z)}{y-z}$$

$$\log(x-y) = \log(y-z) + \log a$$

$$\log(x-y) - \log(y-z) = \log a$$

$$\log\left[\frac{x-y}{y-z}\right] = \log a$$

$\frac{x-y}{y-z} = a$ is one solution.

Taking.

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{y+z-z-x}$$

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{y-x}$$

$$\frac{d(x+y+z)}{2(x+y+z)} = \frac{d(x-y)}{-(x-y)}$$

on integration we get

$$\int \frac{d(x+y+z)}{2(x+y+z)} = -\int \frac{d(x-y)}{x-y}$$

$$\frac{1}{2}\int \frac{2d(x+y+z)}{2(x+y+z)} = -\int \frac{d(x-y)}{x-y}$$

$\frac{1}{2}\log(x+y+z) = -\log(x-y) + \log b$

$\log(x+y+z) = -2\log(x-y) + \log b$

$\log(x+y+z) = -\log(x-y)^{+2} + \log b$

$\log(x+y+z) + \log(x-y)^2 = \log b$

$\log(x+y+z)(x-y)^2 = \log b$

$\log(x+y+z)(x-y)^2 = \log b$

$(x+y+z)(x-y)^2 = b$ is another solution.

$\therefore$ The general solution is $\phi\left[\frac{x-y}{y-z}, (x+y+z)(x-y)^2\right] = 0$

6) Solve: $(x^2-yz)p + (y^2-zx)q = z^2-xy$

solution:

This is of the form $Pp + Qq = R$

The auxillary equation

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \rightarrow (1)$$

Taking the Lagragian multipliers x, y, z (1)

we get each ratio

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2-yz) + y(y^2-zx) + z(z^2-xy)} = 0$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 - xyz + y^3 - xyz + z^2 - xyz} = 0$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz} = 0$$

Also using Lagragian multipliers 1,1,1 in (1)
we get each ratio

$$\frac{dx + dy + dz}{x^2 - yz + y^2 - zx + z^2 - xy} = 0$$

$$\frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx} = 0$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x + y + z} = \frac{d(x+y+z)}{1}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x + y + z} = d(x+y+z)$$

$$\frac{x\,dx + y\,dy + z\,dz}{} = (x+y+z)\,d(x+y+z)$$

Sing we get

$$\int x\,dx + \int y\,dy + \int z\,dz = \int (x+y+z) \int d(x+y+z)$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{(x+y+z)^2}{2}+a$$

$(a+b+c)^3 : (0$

$$\frac{1}{2}(x^2+y^2+z^2)=\frac{(x+y+z)^2}{2}+a$$

$$x^2+y^2+z^2=x^2+y^2+z^2-xy-yz-zx$$

$$x^2+y^2+z^2-x^2-y^2-z^2+xy+yz+zx=0$$

$xy+yz+zx=0$ is one solution.

From ① we have

$$\frac{dx-dy}{x^2-yz-y^2+zx}=\frac{dy-dz}{y^2-zx-z^2+xy}$$

$$\frac{dx-dy}{x^2-y^2+z(x-y)}=\frac{dy-dz}{y^2-z^2+x(y-z)}$$

ie) $$\frac{d(x-y)}{x^2-y^2+z(x-y)}=\frac{d(y-z)}{y^2-z^2+x(y-z)}$$

$x-y+$

ie) $$\frac{d(x-y)}{(x-y)(x+y)+z(x-y)}=\frac{d(y-z)}{(y-z)(z+y)+x(y-z)}$$

ie) $$\frac{d(x-y)}{(x-y)(x+y+z)}=\frac{d(y-z)}{(y-z)(x+y+z)}$$

$$\frac{d(x-y)}{(x-y)\cancel{(x+y+z)}} = \frac{d(y-z)}{(y-z)\cancel{(x+y+z)}}$$

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

$\int$ing we get

$$\int \frac{d(x-y)}{x-y} = \int \frac{d(y-z)}{y-z}$$

$$\log(x-y) = \log(y-z) + \log b$$

$$\log(x-y) - \log(y-z) = \log b$$

$$\log\left[\frac{x-y}{y-z}\right] = \log b$$

$\frac{x-y}{y-z} = b$ is another solution.

$\therefore$ The general solution is $\phi\left[xy+yz+zx, \frac{x-y}{y-z}\right] = 0$

Some standard forms

Type : 1

1) Solve : $pq + p + q = 0$

Solution:

$$pq + p + q = 0$$

$$z = ax + by + c \rightarrow ①$$

Where $ab + b + a = 0$ is a complete integral.

$$ab + a + b = 0$$

$$a + b(a+1) = 0$$

$$b(a+1) = -a$$

$$b = \frac{-a}{a+1}$$

from ①

$\therefore$ The complete integral is $z = ax - (a/a+1)y + c$

$$z = ax - \left(\frac{a}{a+1}\right)y + c \rightarrow ②$$

There is no singular integral.

The general integral put $c = \phi(a)$ in ① we get

$$① \Rightarrow z = ax - \left(\frac{a}{a+1}\right)y + \phi(a) \rightarrow ③$$

$$= \frac{uv' - vu'}{v^2} = a(a+1)$$

Differentiating write to a we get

$$0 = x - \left[\frac{1}{(a+1)^2}\right]y + \phi'(a)$$

Eliminating a from ② and ③ we get the general solution.

STANDARD : 2

1) Solve: $z = px + qy + (q/p)^{-p}$

$$z = px + qy + q/p$$

The given equation of the form

Type : 2

) solve : $z = px + qy + (q/p) - p$

solution:

$$z = px + qy + (q/p) - p \rightarrow ①$$

The given equation of the form

$$z = px + qy + f(p \cdot q)$$

Then put $p = a$; $q = b$

The complete integral of the given equation is

$$① \Rightarrow z = ax + by + b/a - a$$

a and b are constant

To find singular integral

Differential ② write 'b' and 'a' and equate to zero

$$x - b/a^2 - 1 = 0 \rightarrow ②$$

$$x = 1 + b/a^2$$

$$x = \frac{a^2 + b}{a^2}$$

$$y + 1/a = 0 \rightarrow ③$$

$$y = -1/a$$

$a = -1/y$

③ $\Rightarrow a = -1/y$

$x = 1 + b/(-1/y)^2$

$= 1 + b/{+1/y^2}$

$x = 1 + y^2 b$

$y^2 b = x - 1$

$b = \frac{x-1}{y^2}$

sub the value of a and b in ② we get

$z = (-1/y)x + \left(\frac{x-1}{y^2}\right)y + \left(\frac{x-1}{y^2}\right) \times (-y/1) - (-1/y)$

$z = \frac{-x}{y} + \frac{x}{y} - 1/y - \left(\frac{x-1}{y^2}\right)y + 1/y$

$z = 1/y - x/y$

$z = \frac{1-x}{y}$

$yz = 1 - x$

To find general integral

put $b = \phi(a)$ in ② we get

$z = ax + \phi(a)y + \phi(a)/a = a \rightarrow$ ④

Hence the given equation becomes

$$4(1+z^3) = 9az^4 (dz/du)^2 \qquad p = dz/du$$

$$q = dz/du$$

$$4(1+z^3) = 9az^4 (dz/du)^2$$

$$\frac{4(1+z^3)}{9az^4} = \left(\frac{dz}{du}\right)^2$$

squaring on both side

$$\left(\left(\frac{dz}{du}\right)^2\right)^{\frac{1}{2}} = \frac{2}{\sqrt{a}}\left[\frac{\sqrt{1+z^3}}{3z^2}\right]$$

$$\frac{dz}{du} = \frac{2}{\sqrt{a}}\left[\frac{\sqrt{1+z^3}}{3z^2}\right]$$

Hence $\frac{3z^2\sqrt{a}}{\sqrt{1+z^3}}\,dz = 2du$

Integrating

$$\sqrt{a}\sqrt{1+z^3} = u+b$$

squaring on both sides

Hence $(\sqrt{a}\sqrt{(1+z^3)})^2 = (u+b)^2$

$$a(1+z^3) = (u+b)^2 \qquad u = x+ay$$

$$a(1+z^3) = (x+ay+b)^2$$

$\therefore$ The complete integral is $a(1+z^3) = (x+ay+b)^2$

Differentiating the complete integral write to a and b we get.

Differentiatin ②

(1) $(1+z^3) = 2(x+ay+b)(y)$

$$1+z^3 = 2y(x+ay+b)$$

Differentiating write to b we get

$$0 = 2(x+ay+b)(1)$$

$$2(x+ay+b) = 0$$

Hence $1+z^3=0$ which is the sigular solution.

∴ The general solution is founded as usual.

Type : 4

Equations of the form $f_1(x,p) = f_2(y,q)$

1) solve : $q-p = y-x$

solution $q-p = y-x$

The given equation can be rewriten as

$$p-x = q-y.$$

put $p-x = a$

$$p = x+a$$

put $q-y = a$

Differential ④ with write to 'a' the equate to zero

$$0 = x + \phi'(a) y + \frac{a\phi'(a) - \phi(a)}{a^2} - 1 \rightarrow ⑤$$

Eliminate ④ and ⑤ we get general

2) Solve : $z = px + qy - 2\sqrt{pq}$

solutions

$$z = px + qy - 2\sqrt{pq} \rightarrow ①$$

The given equation of the form

$$z = px + qy + f(p.q)$$

Then put $p = a$ $q = b$

The complete integral of the given equat is

$$① \Rightarrow z = ax + by - 2\sqrt{ab} \rightarrow ②$$

a and b are constant

To find singular integral

Differential ② write b and a equate to zero

we get

$$0 = x - \sqrt{(b/a)}$$

$$0 = y - \sqrt{(a/b)}$$

$$x = \sqrt{b/a}$$

$$y = \sqrt{a/b}$$

$\therefore xy = 1$

which is the singular solution

To find the general integral put $b = \phi(a)$ in ①

we get

$$①\Rightarrow z = ax + \phi(a)y - 2\sqrt{a\phi(a)} \rightarrow ③$$

Differentiating write to a we get

$$0 = x + \phi'(a)y - [a\phi(a)]^{-1/2}[a\phi'(a) + \phi(a)]$$

$$x + \phi'(a)y - [a\phi(a)]^{-1/2}[a\phi'(a) + \phi(a)] = 0 \rightarrow$$

Eliminating a from ② and ③ we get the general solution.

Type : 3

equations of the form $f(z, p, q) = 0$

1) solve: $4(1+z^3) = 9z^4pq$

$$4(1+z^3) = 9z^4pq$$

put $z = F(x+ay) = F(u)$

Formula:

$$\therefore p = \frac{dz}{du}\cdot\frac{du}{dx} = \frac{dz}{du}$$

$$q = \frac{dz}{du}\cdot\frac{du}{dy} = \frac{dz}{du}a$$

formula:

$\therefore dz = p\,dx + q\,dy$ $\quad p = x+a$

$q = y+a$

$dz = (x+a)dx + (y+b)dy$

$\therefore dz = (x+a)dx + (y+b)dy + b$: formula

$z = \int g_1(x,a)dx + \int g_2(y,a)dy + b$

$\therefore$ on integration we get $2z = (x+a)^2 + (y+b)^2 + b$

which is the complete solution.

Differentiating the complete integral write to a we get

$1+z^3 = 2(x+ay+b)y$

Differentiating the complete integral write to b we get

$2(x+ay+b)(1) = 0$

$2x+ay+b = 0$

Hence $1+z^3 = 0$ which is the sigular solution.

$\therefore$ The general solution is found as usual.

Type: 4

1\. solve: $q - p = y - x$

solutions

$q - p = y - x$

The given equation can be rewritten as

$p - x = q - y$

put $p-x=a$

$$\boxed{p=x+a}$$

put $q-y=a$

$$\boxed{q=y+a}$$

$$\therefore dz = p\,dx + q\,dy = (x+a)\,dx + (y+a)\,dy$$

$$dz = (x+a)\,dx + (y+a)\,dy$$

on integraction we get

$$\int dz = \int (x+a)\,dx + \int (y+a)\,dy$$

$$\frac{(x+a)^2}{2} + \frac{(y+a)^2}{2}$$

$$2z = (x+a)^2 + (y+a)^2 + b$$ which is the complete solution.

Differentiating the complete solution is write to b we get $0=1$ which is absurd.

Hence there is no sigular integral.

To find there is the general integral put $b=\phi a$ is the complete integral we get

$$2z = (x+a)^2 + (y+a)^2 + \phi(a)$$

Differentiating write to a we get

$$2(x+a) + 2(y+a) + \phi'(a) = 0$$

Eliminating write to a from last two equation we get.

$\therefore$ The general solution

2) problem: 2

solve: $pe^{y} = qe^{x}$

solution:

$$pe^{y} = qe^{x}$$

The given equation can be rewritten.

$$pe^{-x} = qe^{-y}$$

Let $pe^{-x} = a$

$$qe^{-y} = a$$

Hence $p = ae^{x}$ and

$$q = ae^{y}$$

$$\therefore dz = pdx + qdy$$

$$\therefore dz = ae^{x}dx + qe^{y}dy$$

$$z = \int g_1(x, a)dx + \int g_1(y, a)dy + b$$

$\therefore z = a(e^{x} + e^{y}) + b$ which is the complete integral

we can easily see that there is no sigular integral.

$\therefore$ The general integral can be found as usual.

Charpit's methods $f(x, y, z, p, q)$

Find the complete integral of $px+qy=pq$

$$px+qy=pq$$

Let $f(x,y,z,p,q) = px+qy-pq \rightarrow (1)$

$\therefore f_p = x-q \qquad f_y = q$

$f_q = y-p \qquad f_z = 0$

$f_x = p$

The charpit's auxillary equation is

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\therefore \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{dz}{0}$$

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-px+pq-qy+pq} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{dz}{0}$$

Taking

$$\frac{dp}{p} = \frac{dq}{q}$$

$\int$ ing

$$\int \frac{dp}{p} = \int \frac{dq}{q} \text{ we get}$$

$$\log p = \log q + \log a$$

$$\log p = \log qa$$

$p = aq \rightarrow (2)$

subsitituting (2) in (1) we get

$$aqx + qy - aq^2 = 0$$

$$\therefore aq^2 = q(ax+y)$$

$$q^2 = \frac{q(ax+y)}{a}$$

$$q = \frac{(ax+y)}{a} \qquad qa = ax+y$$

$$\therefore p = aq$$

from (2) we get

$$p = ax+y \rightarrow (3)$$

$$dz = pdx + qdy$$

$$dz = (ax+y)dx + \left(\frac{ax+y}{a}\right)dy$$

$$adz = a(ax+y)dx + (ax+y)dy$$

$$= a^2 x dx + ay dx + ax dy + y dy$$

$$adz = a^2 x dx + a(ydx + xdy) + ydy$$

$$adz = a^2 x dx + a\, d(xy) + ydy$$

on integraction we get

$$\int adz = a^2 \int x dx + a \int d(xy) + \int y dy$$

$$az = \frac{a^2 x^2}{2} + axy + \frac{y^2}{2} + b$$

$$az = \frac{a^2x^2 + 2axy + y^2}{2} + b$$

$az = \frac{1}{2}(y+ax)^2 + b$ which is the

4 2) Find the complete integral for $z = px + qy + p^2 + q^2$

solutions:

$$z = px + qy + p^2 + q^2$$

Let $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 \rightarrow ①$

$f_p = -x - 2p$ $\qquad f_y = -q$

$f_q = -y - 2q$ $\qquad f_z = 1$

$f_x = -p$

The charpit's auxillary equation are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{f_z}$$

$$\frac{dp}{-p + p} = \frac{dq}{-q + q} = \frac{dz}{-p(-x-2p) - q(-y-2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q} = \frac{dz}{-1}$$

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{+px + 2p^2 + qy + 2q^2} = \frac{dx}{x+2p} = \frac{dy}{y+2q} = \frac{dz}{1}$$

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{px + qy + 2p^2 + 2q^2} = \frac{dx}{x+2p} = \frac{dy}{y+2q} = \frac{dz}{1}$$

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p)} = \frac{dx}{x+2p} = \frac{dy}{y+2q} = \frac{dz}{1}$$

Taking $\frac{dp}{0} = \frac{dq}{0}$ we get

$dq = 0$ $\qquad dp = 0$

$\int$ing $\qquad \int dp = 0$

$\int dq = 0$ $\qquad p = a$

$$q = b \quad ; \; p = a$$

substituting in ① we get

$$z = ax + by + a^2 + b^2$$ which is the complete integral

(8) Find the complete integrals of $q = px + p^2$

solutions:

$$q = px + p^2$$

Let $f(x, y, z, p, q) = q - px - p^2 \rightarrow ①$

$\therefore f_p = -x - 2p$

$f_q = 1 \qquad f_y = 0$

$f_x = -p \qquad f_z = 0$

The charpit's auxilliary equation is

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-f_z}$$

$$\frac{dp}{-p+0} = \frac{dq}{0+0} = \frac{dz}{-p(-x-2p) - q(1)} = \frac{dx}{-(-x-2p)} = \frac{dy}{-1} = \frac{dz}{0}$$

$$\frac{dp}{-p} = \frac{dq}{0} = \frac{dz}{-p(-x-2p) - q} = \frac{dx}{x+2p} = \frac{dy}{-1} = \frac{dz}{0}$$

The second relation gives

$$dq = 0$$

$\int$ ing

$$\int dq = 0$$

$$q = a \rightarrow (2)$$

substituting (2) in (1) we get

$$p^2 + px - a = 0$$

solving for p we get

$$p^2 + px - a = 0$$

$a = 1$
$b = x$
$c = a$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-x \pm \sqrt{x^2 - 4a}}{2}$$

$$p = \frac{1}{2}\left[-x \pm \sqrt{x^2 - 4a}\right]$$

Now $dz = p\,dx + q\,dy$

$$dz = \frac{1}{2}\left[-x \pm \sqrt{x^2 - 4a}\right]dx + a\,dy$$

$$dz = -\frac{1}{2}x\,dx \pm \sqrt{x^2 - 4a}\,dx + a\,dy$$

on integration we get

$$\int dz = -\frac{1}{2}\int x\,dx \pm \int \frac{\sqrt{x^2 - 4a}}{2}\,dx + \int a\,dy$$

$$z = -\frac{1}{2}\frac{x^2}{2} \pm \frac{x}{2}\sqrt{x^2 - 4a} + \frac{4a}{2}\sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right) + ay + b$$

4) solve: $(p^2+q^2)y = qz$

solution:

$$(p^2+q^2)y = qz$$

$$p^2y + q^2y = qz$$

Here

$$F(x,y,z,p,q) = (p^2+q^2)y - qz = 0 \rightarrow (1)$$

$F_x = 0$ $\quad$ $F_p = 2py$

$F_y = p^2+q^2$

$F_q = 2qy - z$

$F_z = -q$

The Auxillary equation is

$$\frac{dp}{F_x + pF_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0}$$

$$\frac{dp}{0+p(-q)} = \frac{dq}{p^2+q^2+q(-q)} = \frac{dz}{-p(2py) - q(2py-z)}$$

$$= dx/-2py = \frac{dy}{-2qy+z} = \frac{df}{0}$$

$$\frac{dp}{-pq} = \frac{dq}{p^2+q^2-q^2} = \frac{dz}{-2pqy - 2pqy + qz} = \frac{dx}{-2py}$$

$$= \frac{dy}{-2qy+z} = \frac{df}{0}$$

consider 1^{st} 2 ratios

$$\frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\frac{dp}{-q} = \frac{dq}{p}$$

$$dp/q = -dq/p$$

$$p\,dp + q\,dq = 0$$

Sing

$$\int p\,dp + \int q\,dq = 0$$

$$\frac{p^2}{2} + \frac{q^2}{2} = \frac{a^2}{2}$$

$$p^2 + q^2 = a^2 \rightarrow (2)$$

substituting (2) in (1)

from (1) $\Rightarrow (p^2+q^2)y = qz$

$$a^2 y = qz$$

$$q = \frac{a^2 y}{z} \rightarrow (3)$$

substituting (3) in equation (2)

(2) $\Rightarrow p^2 + q^2 = a^2$

$$p^2 + \left(a^2 y/z\right)^2 = a^2$$

$$p^2 = a^2 - \frac{a^4 y^2}{z^2}$$

$$p^2 = \frac{a^2 z^2 - a^4 y^2}{z^2}$$

$$p^2 = a^2 \left[\frac{z^2 - a^2 y^2}{z^2}\right]$$

squaring on both sides

$$p = \frac{a\sqrt{z^2 - a^2y^2}}{z} \rightarrow ④$$

subsutituting ③ and ④ in differential relation

$$dz = p\,dx + q\,dy$$

$$dz = \left[\frac{a\sqrt{z^2 - a^2y^2}}{z}\right] dx + \left(a^2y/z\right) dy$$

$$dz = 1/z\left[a\sqrt{z^2 - a^2y^2}\right] dx + a^2y\,dy$$

$$z\,dz = \left[a\sqrt{z^2 - a^2y^2}\right] dx + a^2y\,dy$$

$$z\,dz - a^2y\,dy = \left(a\sqrt{z^2 - a^2y^2}\right) dx$$

$$\frac{z\,dz - a^2y\,dy}{\sqrt{z^2 - a^2y^2}} = a\,dx$$

$$2/2 \cdot \frac{z\,dz - a^2y\,dy}{\sqrt{z^2 - a^2y^2}} = a\,dx$$

$$d\left(\sqrt{z^2 - a^2y^2}\right) = a\,dx$$

Sing

$$\int d\left(\sqrt{z^2 - a^2y^2}\right) = a\int dx$$

$$\sqrt{z^2 - a^2y^2} = ax + b$$

squaring on both sides

$$z^2 - a^2y^2 = (ax+b)^2 \rightarrow (5)$$

This is the requried complete integral

Differentiating (5) write a and b we get

$$-2ay^2 = 2(ax+b)\cdot(x)$$

$\Rightarrow$

$$2x(ax+b) + 2ay^2 = 0$$

$b \Rightarrow$ $$2(ax+b)(1) = 0$$

$$2(ax+b) = 0$$

using the last two equation in the complete integral we get $z = 0$

which is the singular solution integral.

Now, To find the general integral put $b = \phi(a)$ in (5) we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2$$

Differentiating write to a we get

$$-2ay^2 = 2[[ax + \phi(a)][x + \phi'(a)]] \rightarrow (6)$$

$\therefore$ General integrating is obtained by eliminatin a from (5) and (6)

5) solve: $pxy + pq + qy - yz = 0$

solution:

$$pxy + pq + qy - yz = 0$$

Let $f(x, y, z, p, q) = pxy + pq + qy - yz \rightarrow ①$

$f_z = -y$

$f_p = xy + p$; $f_q = p + y$; $f_x = py$; $f_y = px + q - z$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{df}{0}$$

$$\frac{dp}{py + p(-y)} = \frac{dq}{px + q - z + q(-y)} = \frac{dz}{-p(xy + q) - q(p + y)}$$

$$= \frac{dx}{-(xy + q)} = \frac{dy}{-(p + y)} = \frac{df}{0}$$

$$\frac{dp}{py - py} = \frac{dq}{px + q - z - qy} = \frac{dz}{-pxy - pq - qp - qy}$$

$$= \frac{dx}{-xy - q} = \frac{dy}{-p - y} = \frac{df}{0}$$

consider the first two ratio's

$$\frac{dp}{py - py} = \frac{dp}{0}$$

$dp = 0$

Jing we get

$\int dp = 0$

$p = a$ (constant) $\rightarrow ②$

From ①

$$pxy + pq + qy - yz = 0$$

$$pq + qy = yz - pxy \qquad \therefore p = a$$

$$q(p+y) = yz - axy$$

put $p = a$

$$q(a+y) = yz - axy$$

$$q = \frac{yz - axy}{a+y} \rightarrow ③$$

The differential equation is

$$dz = pdx + qdy \rightarrow ④$$

sub ② and ③ in equation ④

$$④ \Rightarrow \quad dz = pdx + qdy$$

$$dz = adx + \left[\frac{yz - axy}{a+y}\right] dy$$

$$dz = adx + y\left[\frac{z - ax}{a+y}\right] dy$$

$$dz - adx = y\left[\frac{z - ax}{a+y}\right] dy$$

$$\frac{dz - adx}{z - ax} = \frac{y}{a+y} dy$$

$$\frac{d(z - ax)}{z - ax} = \frac{y + a - a}{a+y} dy$$

$$\frac{d(z - ax)}{z - ax} = \left[\frac{y+a}{a+y} - \frac{a}{a+y}\right] dy$$

$$\frac{d(z-ax)}{z-ax} = (1 - a/y+a)\,dy$$

Sing we get

$$\int \frac{d(z-ax)}{z-ax} = \int (1 - a/y+a)\,dy$$

$$\log(z-ax) = (y-a)\log(y+a) \mp b$$

$$\log(z-ax) + y - a\log(y+a) \pm b$$

$$\log(z-ax) + a\log(y+a) = y+b$$

$$\log(z-ax) + \log(y+a)^a = y+b$$

$$\log(z-ax)(y+a)^a = y+b$$

$$(z-ax)(y+a)^a = e^y + b$$

$(z-ax)(y+a)^a = be^y$ is the compelete solution.

The single integral and the general integral can be founded as usual.

6) Deduce the complete integral of the standar form $z = px + qy + f(p,q)$ from charpit's method

$$z = px + qy + f(p,q)$$

charpit's auxillary equation aregiven by

$$dp/0 = \frac{dq}{0} = \cdots$$

$dp = 0$

$\int$ ing we get

$\int dp = 0$

$p = a$

$dq = 0$

$\int$ ing we get

$\int dq = 0$

$q = b$

Hence the complete integral is $z = ax + by + f(a, b)$

Unit - 6

Applications at differential equation finding

orthogonal trajectories:

Definition:

orthogonal trajectories

A trajectory of a family of curves is a curve cutting all the member of the system according to some law

For example:

A curve cutting a family at curves at a constant angle is a trajection. If this constant angle be a right angle then it is known as orthogonal trajectories.

Families of curves related in this way occur in applied problems.

In two dimensional problems in the flow of that heat the curves along which the heat flow takes place at the same temperature are orthogonal trajectories. Similarly we can see the orthogonal trajectories in the flow of electricity the flow of water from a lake into a narrow channel.

I Working rule to find orthogonal trajectories.

If the curves are given in cartesian coordinates.

Step 1: Obtain the differential equation of the given family of curves.

Step 2: Replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ in the differential equation obtained in step 1, Also obtain the differential equation of the orthogonal.

Step 3: Obtain the general solution of the differential equation in step 2 which is the required families of orthogonal trajectories

If the curves are given in polar coordinates.

Step 1: Obtain the differential equation of the given family of curves.

Step 2: Replace $r\frac{d\theta}{dr}$ by $\frac{-1}{r}\left(\frac{dr}{d\theta}\right)$

Step 3: Obtain the general solution of the differential equation obtained in step 2

Find the orthogonal trajectories of the family of circles $x^2+y^2=a^2$

Given $x^2+y^2=a^2$

diff w.r to x.

$$2x+2y\frac{dy}{dx}=0$$

$$2x=-2y\frac{dy}{dx}$$

$$-\frac{x}{y}=\frac{dy}{dx}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$-\frac{dx}{dy}=-\frac{x}{y}$$

$$\frac{dx}{x}=\frac{dy}{y}$$

Integrating

$$\log x=\log y+\log m.$$

$x=ym$ is the family of the curve

Find the orthogonal trajectories of the system of curve $\left(dy/dx\right)^2=a/x$.

$$\left(\frac{dy}{dx}\right)^2=\frac{a}{x}$$

$$\frac{dy}{dx}=\frac{\sqrt{a}}{\sqrt{x}}$$

Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{\sqrt{a}}{\sqrt{x}}$$

$$-\sqrt{x}\, dx = \sqrt{a}\, dy.$$

Integrating

$$\frac{1}{\sqrt{a}}\left(\frac{x^{3/2}}{3/2}\right) = y+m$$

$$\frac{2x^{3/2}}{3\sqrt{a}} = y+m$$

$\frac{4x^3}{9a} = (y+m)^2$ is the system of orthogonal trajectories.

Find the orthogonal trajectories at the family at curves $r = a\sin\theta$

Given $r = a\sin\theta \longrightarrow (1)$

Diff w.r to θ

$$\frac{dr}{d\theta} = a\cos\theta \longrightarrow (2)$$

(1) divides (2)

$$(1)/(2) \Rightarrow r\Big/\frac{dr}{d\theta} = \frac{a\sin\theta}{a\cos\theta}.$$

$$r\left(\frac{d\theta}{dr}\right) = \tan\theta.$$

Replace $r\left(\frac{d\theta}{dr}\right)$ by $\left(\frac{-1}{r}\right)\left(\frac{dr}{d\theta}\right)$

$$\frac{-1}{r}\left(\frac{dr}{d\theta}\right) = \tan\theta$$

$$-\frac{dr}{r} = \tan\theta . d\theta$$

Integrating

$$-\log r = \log \sec\theta + \log m$$

$$\log\left(1/r\right) = \log(m \sec\theta)$$

$$\frac{1}{r} = m\sec\theta$$

$$r = \frac{1}{m}\,\frac{1}{\sec\theta} \qquad c = \frac{1}{m}$$

$r = c\cos\theta$ is the family of orthogonal trajectories.

Find the equation of system of orthogonal trajectories of the parabola $r = \frac{2a}{1+\cos\theta}$.

$$r = \frac{2a}{1+\cos\theta} \longrightarrow (1)$$

Diff w r to (1)

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1+\cos\theta}(\sin\theta)$$

$$\frac{1}{r}\left(\frac{dr}{d\theta}\right) = \frac{\sin\theta}{1+\cos\theta}$$

$$r\frac{d\theta}{dr} = \frac{2\cos^2\theta/2}{\cancel{2\cos\theta/2}\ \sin\theta/2}$$

$$\Rightarrow \frac{\quad}{dr} = \cot \theta/2$$

Replace $\Rightarrow \frac{d\theta}{dr}$ by $\frac{-1}{r}\left(\frac{d\theta}{dr}\right)$

$$\frac{-1}{r}\left(\frac{d\theta}{dr}\right) = \cot \theta/2$$

$$\frac{-dr}{r} = \cot \theta/2 \cdot d\theta .$$

Integrating

$$-\log r = \frac{\log \sin \theta/2}{1/2} + \log m$$

$$-\log r = 2 \log \sin \theta/2 + \log m .$$

$$\frac{1}{r} = \sin \theta/2 \,(m)^2 .$$

$$r = \frac{1}{\sin \theta/2}\left(\frac{1}{m}\right) .$$

$$r = \frac{1}{\left(\frac{1-\cos 2\theta}{2}\right)}\,(1/m)$$

$r = \dfrac{2c}{1-\cos 2\theta}$ is the family of orthogonal trajectories.

5. Find the orthogonal trajectories for $x^{2/3} + y^{2/3} = a^{2/3}$.

Given eqn is $x^{2/3}+y^{2/3}=a^{2/3}$ —(1)

Diff w.r.t. x

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

$$x^{-1/3}+y^{-1/3}\frac{dy}{dx}=0$$

$$y^{-1/3}\frac{dy}{dx}=-x^{-1/3}$$

$$\frac{dy}{dx}=\frac{-x^{-1/3}}{y^{-1/3}}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$-\frac{dx}{dy}=\frac{-x^{-1/3}}{y^{-1/3}}$$

$$\frac{dx}{x^{-1/3}}=\frac{dy}{y^{-1/3}}$$

$$x^{1/3}.dx=y^{1/3}.dy$$

Integrating

$$\frac{x^{4/3}}{4/3}=\frac{y^{4/3}}{4/3}+m$$

$$\frac{3x^{4/3}}{4}=\frac{3y^{4/3}}{4}+m$$

Find the orthogonal trajectories for $y^2 = 4ax$

Given $y^2 = 4ax \longrightarrow (1)$

Diff w.r.t to x

$$2y \frac{dy}{dx} = 4a$$

$$y \frac{dy}{dx} = 2a \longrightarrow (2)$$

$(1) \div (2)$

$$\frac{y^2}{y(dy/dx)} = \frac{4ax}{2a}$$

$$\frac{y}{dy/dx} = 2x$$

$$\frac{dy}{dx} = y/2x$$

Replace $-\frac{dx}{dy} = \frac{dy}{dx}$

$$-\frac{dx}{dy} = \frac{y}{2x}$$

$$-2x \cdot dx = y \cdot dy.$$

Integrating

$$-\frac{2x^2}{2} = \frac{y^2}{2} + m$$

$y^2 + x^2 + m = 0$ is the family of orthogonal trajectories

7. Find the orthogonal trajectories for

$r = a\cos\theta$

Given $r = a\cos\theta \rightarrow (1)$

Diff w.r to θ

$$\frac{dr}{d\theta} = -a\sin\theta \rightarrow (2)$$

(1) ÷ (2)

$$\frac{r}{dr/d\theta} = \frac{a\cos\theta}{-a\sin\theta}$$

$$r\left(\frac{d\theta}{dr}\right) = -\cot\theta$$

Replace $r\left(\frac{d\theta}{dr}\right)$ by $-\frac{1}{r}\frac{dr}{d\theta}$

$$-\frac{1}{r}\frac{dr}{d\theta} = -\cot\theta$$

$$\frac{dr}{r} = \cot\theta \cdot d\theta .$$

Integrating

$$\log r = \log\sin\theta + \log m .$$

$$r = \sin\theta\, m .$$

$$r = m\sin\theta .$$

8 Find the orthogonal trajectories for

$r^n = a^n \cos n\theta$

Given $r^n = a^n \cos n\theta$

Taking log on both sides.

$$\log r^n = \log(a^n \cos n\theta)$$

$$n\log r = \log a^n + \log \cos n\theta$$

diff w.r to θ

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{1}{\cos n\theta}(-\sin n\theta)(n)$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

$$r\frac{d\theta}{dr} = -\cot n\theta$$

Replace $r\frac{dr}{d\theta} = -\cot n\theta$.

$$\frac{dr}{r} = \cot n\theta \, d\theta$$

Integrating

$$\log r = 1/n \log \sin n\theta + \log c$$

9. Find the orthogonal & for $y^2 = 4a(x+a)$

Given that $y^2 = 4a(x+a) \rightarrow ①$.

diff w r to x

$$2y\frac{dy}{dx} = 4a \rightarrow ②$$

$$\frac{2y \; dy/dx}{4} = a$$

$$① \Rightarrow y^2 = \left(2y \cdot \frac{dy}{dx}\right)(x+5)$$

$$y^2 = \left(2y\frac{dy}{dx}\right)\left(x + \frac{xy\frac{dy}{dx}}{x}\right)$$

$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y\frac{dy}{dx}}{e}\right)$$

$$y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

divide by y

$$y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2 \rightarrow ③$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$y = 2x\left(-\frac{dx}{dy}\right) + y\left(-\frac{dx}{dy}\right)^2$$

$$y = -2x\frac{dx}{dy} + y\left(\frac{dx}{dy}\right)^2$$

from ③ & ④.

$y^2 = 2a(x+a)$ is self orthogonal.

The Brachistochrone problem:.

Suppose A and B are two points in space but not in the same vertical one at A and B be connected by a plane curve C and A is at highest level.

Consider a smooth particle which falls from A to B along C. The problem shape of c is called Brachistochrone problem

Differential equation of Brachistochrone problem:

Consider the Brachistochrone problem. choose the horizontal and vertical lines through A as x-axis and y-axis respectively. let P(x,y) be the position of the particle at time t.

let 's' be the length "AP" let α be the angle made by the tangent to P with y axis

let v be the velocity of the particle at P

Then by the principle of conservation of energy we have.

$$\frac{1}{2} m v^2 - mgy = 0$$

$$\frac{1}{2} m v^2 = mgy$$

$$\frac{v^2}{2} = gy$$

$$v^2 = 2gy$$

$$v = \sqrt{2gy} \rightarrow ①$$

By Snell's law

$$\frac{\sin \alpha}{v} = c \longrightarrow (2)$$

By the equation of calculus.

$$\sin \alpha = \frac{1}{\sqrt{1+(dy/dx)^2}}$$

$$(2) \Rightarrow \quad cv = \frac{1}{\sqrt{1+(dy/dx)^2}}$$

$$cv^2 = \frac{1}{1+(dy/dx)^2}$$

$$(5) \Rightarrow 2gy = \frac{1}{1+(dy/dx)^2}$$

$$y(1+(dy/dx)^2) = \frac{1}{2g} = k \text{ (constant)}$$

$y(1+(dy/dx)^2) = k$ this is the differential equation of brachistochrone problem.

Solution of Brachistochrone problem.

We have

$$y\left(1+(dy/dx)^2\right) = k.$$

$$1+(dy/dx)^2 = \frac{k}{y}.$$

$$(dy/dx)^2 = \frac{k}{y} - 1$$

$$\frac{dy}{dx} = \sqrt{\frac{k-y}{y}}$$

$$dy = \sqrt{\frac{k-y}{y}}\, dx$$

$$\sqrt{\frac{y}{k-y}}\, dy = dx$$

Put $\frac{y}{k-y} = \tan^2\phi$

$$y = (k-y)\tan^2\phi$$

$$y = k\tan^2\phi - y\tan^2\phi$$

$$y(1+\tan^2\phi) = k\tan^2\phi$$

$$y\sec^2\phi = k\tan^2\phi$$

$$y\left(\frac{1}{\cos^2\phi}\right) = k\left(\frac{\sin^2\phi}{\cos^2\phi}\right)$$

$$y = k\sin^2\phi$$

$$dy = k\, 2\sin\phi\cos\phi\, d\phi$$

Now, $dx = \sqrt{y/k-y}\; dy$

$$= 2k\sin\phi\cos\phi\, d\phi$$

$$= 2k\left(\frac{1-\cos 2\phi}{2}\right) d\phi$$

$$dx = k(1-\cos 2\phi)\, d\phi$$

Integrating

$$x = k\left(\phi - \frac{\sin 2\phi}{2}\right) + c$$

$$x = \frac{k}{2}(2\phi - \sin 2\phi) + c$$

when $c = 0$

$$x = \frac{k}{2}(2\phi - \sin 2\phi)$$

Also $y = k \sin^2\phi$

$$y = \frac{k}{2}\left(\frac{1-\cos 2\phi}{2}\right)$$

$$y = \frac{k}{2}(1-\cos 2\phi)$$

Put $a = k/2$ and $\theta = 2\phi$ in (x, y)

$$x = a(\theta - \sin\theta)$$

$$y = a(1-\cos\theta)$$

These are the parametric eqn of cycloid.